AN ANALYSIS OF THE SEQUENCE $X_{n+2} = i\, m\, X_{n+1} + X_n$

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ABSTRACT. We analyze the sequence $X_{n+2} = imX_{n+1} + X_n$, with $X_1 = X_2 = 1 + i$, where i is the imaginary number and m is a real number. Plotting the sequence in the complex plane for different values of m, we see interesting figures from the conic sections. For values of m in the interval $(-2, 2)$ we show that the figures generated are ellipses. We also provide analysis which prove that for certain values of m, the sequence generated is periodic with even period.

1. Introduction and Notation

The sequence 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$ is the famous Fibonacci sequence which has the same two starting values and a number in the sequence is generated by adding the previous two consecutive ones [1], [2]. The sequence analyzed in this paper gets its general idea from the Fibonacci sequence. We start with the same two numbers and use a variation of the idea of generating the next number via modifying the addition of previous two consecutive numbers. The sequence studied in this paper is of the form:

$$X_1 = X_2 = 1 + i$$

and

$$X_{n+2} = i\, m\, X_{n+1} + X_n$$

where: m is any real number, i is the imaginary number $\sqrt{-1}$, and $n \geq 1$. This sequence has interesting properties that we present. Unlike the Fibonacci sequence which grows without bound, this sequence has different behavior depending on the value assigned to m.

2. The Sequence and Conditions on m

To begin the analysis of our sequence, we first determine the conditions on m that changes the behavior of the sequence. The recursive sequence is analyzed using concepts from difference equations [3]. This is a second order, constant coefficient difference equation and its characteristic equation is

$$r^2 = imr + 1$$

Using the quadratic formula we solve the equation for r and get the two roots $r_{1,2}$ in terms of m. These roots help determine the conditions we set on m.

$$r_{1,2} = \frac{m}{2}\, i \pm \frac{\sqrt{4-m^2}}{2}$$

We get the following three cases depending on the value of the expression $4 - m^2$.

(i) $4 - m^2 = 0$, results in repeated roots $r_{1,2} = \frac{m}{2}\, i$ that are purely imaginary

(ii) $4 - m^2 > 0$, results in two distinct complex roots

(iii) $4 - m^2 < 0$, results in distinct purely imaginary roots.

3. Periodicity for $-2 < m < 2$

Since $-2 < m < 2$, the substitution $m = 2\sin\theta$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ retains the condition on m and allows us to rewrite the roots as $r_{1,2} = \pm\cos\theta + i\sin\theta$. Thus

$$r_1 = \cos\theta + i\sin\theta \quad \text{and} \quad r_2 = -\cos\theta + i\sin\theta.$$

The general solution to the difference equation is of the form;

$$X_n = C_1\, r_1^n + C_2\, r_2^n$$

where $r_{1,2}$ can also be written in the following form

$$r_1 = e^{i\theta}$$
$$r_2 = e^{i(\pi-\theta)}.$$

Substituting for $r_{1,2}$, the general solution takes the form $X_n = C_1 e^{in\theta} + C_2 e^{in(\pi-\theta)}$.

Theorem 3.1. *If θ is a rational multiple of a revolution, then the sequence X_n is periodic and has an even period.*

Proof. The sequence X_n is periodic when $X_{n+k} = X_n$ for some integer k. The smallest integer k that satisfies the condition is called the period of the sequence.

$$X_{n+k} = C_1\, e^{i(n+k)\theta} \;+\; C_2\, e^{i(n+k)(\pi-\theta)}$$
$$X_{n+k} = C_1\, e^{in\theta}\, e^{ik\theta} \;+\; C_2\, e^{in(\pi-\theta)}\, e^{ik(\pi-\theta)}$$

So, $X_{n+k} = X_n$, when both of the equations $e^{ik\theta} = 1$ and $e^{ik(\pi-\theta)} = 1$ are satisfied.

From the first condition, $k\theta = 2\ell\pi$ and from the second, $k(\pi-\theta) = 2j\pi$ for some integers ℓ and j.

We can solve the first equation to find θ. From the second equation we get

$$k\pi - k\theta = 2j\pi$$
$$k\pi = 2j\pi + 2\ell\pi$$

This shows that for the sequence to be periodic, k has to be even and that $\theta = \dfrac{2\ell\pi}{k}$.

4. Plotting the Sequence in the Complex Plane

In this section the sequence X_n of complex numbers are plotted on the complex plane for various values of m in the interval $(-2, 2)$. In **Figure 1**, we chose $m = 1.17$ and the number of values of the sequence drawn is 200. Notice that this sequence is not periodic. **Figure 1b)** is the overlay of two ellipses. As we plot more and more points of the sequence, the gaps on the ellipses shown in **Figure 1b)** gets smaller.

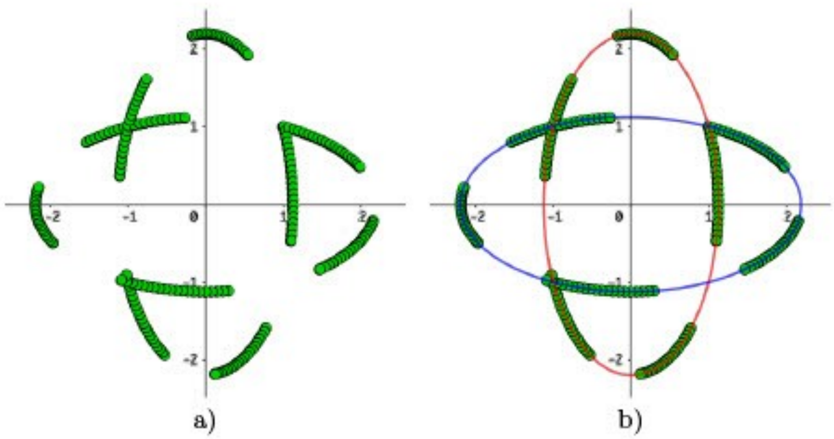

Figure 1. The case where $m = 1.17$

In **Figure 2**, we pick $\ell = 1$ and $k = 20$, resulting in $m = 2 \sin \frac{2\pi}{20}$ and the period is 20. Just like the non periodic case, in **Figure 2b** we observe that the points of the sequence are on two ellipses whose axes are perpendicular to each other. Notice here that 4 points where the two ellipses intersect are repeated twice.

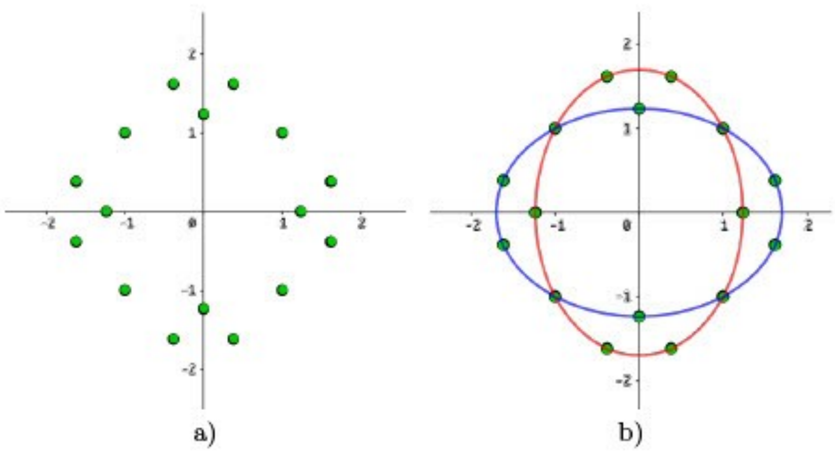

Figure 2. The case where $\ell = 1$ and $k = 20$

5. The Equation of the Ellipse for $m = 1$

Below we solve for an explicit representation of the two ellipse equations for the case when $X_1 = X_2 = 1 + i$ and $m = 1$. The difference equation we solve here is:

$$X_{n+2} = i\, X_{n+1} + X_n.$$

Substituting $X_n = r^n$, we get the characteristic equation

$$0 = r^2 - ir - 1.$$

This gives us the two roots $r_{1,2} = i \pm \dfrac{\sqrt{3}}{2}$. Thus we get, $r_1 = e^{i\frac{\pi}{6}}$ and $r_2 = e^{i\frac{5\pi}{6}}$. Substituting the roots in the general equation, we get

$$X_n = C_1 e^{i\frac{n\pi}{6}} + C_2 e^{i\frac{5n\pi}{6}}$$

Next, we solve for the coefficients C_1 and C_2 by writing the matrix equation and using Cramer's rule to solve them.

$$\begin{bmatrix} i+1 \\ i+1 \end{bmatrix} = \begin{bmatrix} e^{i\frac{\pi}{6}} & e^{i\frac{\pi}{6}} \\ e^{i\frac{\pi}{3}} & e^{i\frac{\pi}{3}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

This results in the solution:

$$C_1 = \frac{\sqrt{3}+1}{\sqrt{3}} \text{ and } C_2 = \frac{\sqrt{3}-1}{\sqrt{3}}.$$

Substituting the coefficients, we get the expression for the sequence. This general form is simplified and separated into real and imaginary parts as follows.

$$X_n = \frac{\sqrt{3}+1}{\sqrt{3}}\, e^{\frac{i\pi n}{6}} + \frac{\sqrt{3}-1}{\sqrt{3}}\, e^{\frac{i5\pi n}{6}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}} \left(\cos\frac{\pi n}{6} + i\, \sin\frac{\pi n}{6}\right) + \frac{\sqrt{3}-1}{\sqrt{3}} \left(\cos\frac{5\pi n}{6} + i\, \sin\frac{5\pi n}{6}\right)$$

$$= \left[\frac{\sqrt{3}+1}{\sqrt{3}} \cos\frac{\pi n}{6} + \frac{\sqrt{3}-1}{\sqrt{3}} \cos\frac{5\pi n}{6}\right] + i\, \left[\frac{\sqrt{3}+1}{\sqrt{3}} \sin\frac{\pi n}{6} + \frac{\sqrt{3}-1}{\sqrt{3}} \sin\frac{5\pi n}{6}\right]$$

To get the equations of the two ellipses, we consider the odd and even values of n separately. First consider the real part of the sequence X_n

$$Re(X_n) = \frac{\sqrt{3}}{\sqrt{3}} \left(\cos\frac{\pi n}{6} + \cos\frac{5\pi n}{6}\right) + \frac{1}{\sqrt{3}} \left(\cos\frac{\pi n}{6} - \cos\frac{5\pi n}{6}\right)$$

Consider the case where n is even, then we can write $n = 2k$ for some integer k

$$Re(X_{2k}) = \left(\cos\frac{\pi k}{3} + \cos\frac{5\pi k}{3}\right) + \frac{1}{\sqrt{3}}\left(\cos\frac{\pi k}{3} - \cos\frac{5\pi k}{3}\right)$$

$$Re(X_{2k}) = 2\cos\frac{\pi k}{3}.$$

Now consider the imaginary part of the sequence X_n

$$Im(X_n) = \frac{\sqrt{3}}{\sqrt{3}}\left(\sin\frac{\pi n}{6} + \sin\frac{5\pi n}{6}\right) + \frac{1}{\sqrt{3}}\left(\sin\frac{\pi n}{6} - \sin\frac{5\pi n}{6}\right).$$

Substituting $n = 2k$

$$Im(X_{2k}) = \left(\sin\frac{\pi k}{3} + \sin\frac{5\pi k}{3}\right) + \frac{1}{\sqrt{3}}\left(\sin\frac{\pi k}{3} - \sin\frac{5\pi k}{3}\right)$$

$$Im(X_{2k}) = \frac{2}{\sqrt{3}}\ \sin\frac{\pi k}{3}.$$

The equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be parameterized as $x = a\cos\alpha$ and $y = b\sin\alpha$. The above calculations give us

$$Re(X_{2k}) = 2\cos\frac{\pi k}{3} \text{ and } Im(X_{2k}) = \frac{2}{\sqrt{3}}\ \sin\frac{\pi k}{3}.$$

Comparing the two, we get $a = 2$, $b = \frac{2}{\sqrt{3}}$ and $\alpha = \frac{\pi k}{3}$.

since the real and imaginary parts yield the parametric form of the ellipse, when written in standard form the equation becomes

$$x^2 + 3y^2 = 4.$$

Similarly we derive the case for n odd, that is, $n = 2k + 1$ for some integer k.

$$Re(X_{2k+1}) = \frac{2}{\sqrt{3}}\cos\frac{\pi k}{3}$$

$$Im(X_{2k+1}) = 2\sin\frac{\pi k}{3}$$

The real and imaginary parts of the odd computations yield the parametric form of the other ellipse. When written in standard form the equation becomes

$$3x^2 + y^2 = 4.$$

In **Figure 3a** is the ellipse when n is even and **Figure 3b** is the ellipse for odd n. **Figure 3c** presents the sequence points and the two superimposed ellipses.

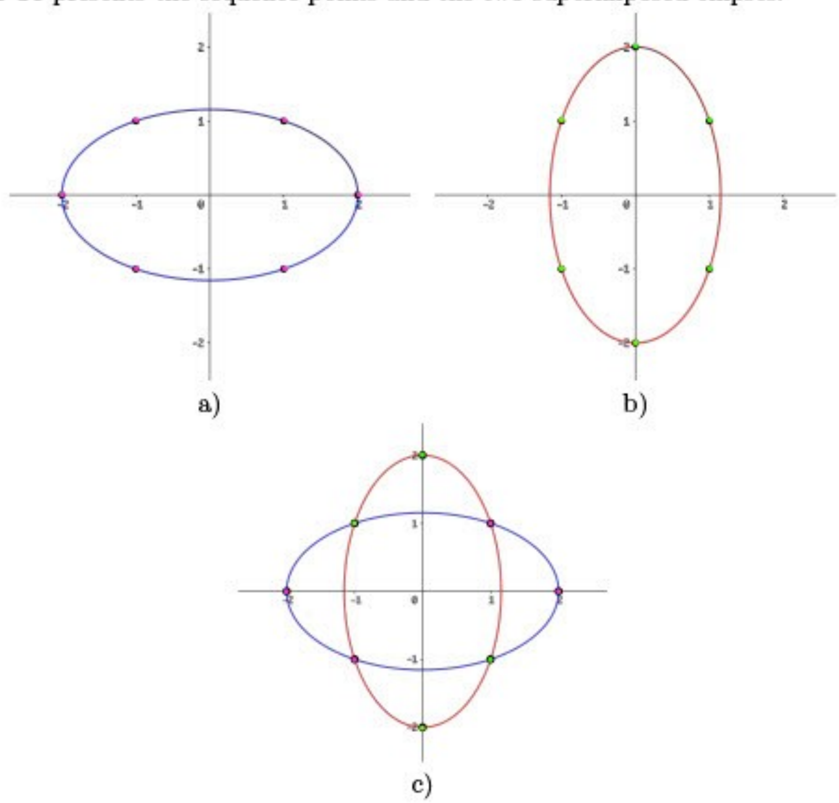

Figure 3. The case where $\ell = 1$ and $k = 6$ a) n is even ; b) n is odd; c) n is any integer

6. Conclusion and Further Study

We observed that for $-2 < m < 2$, the sequence is bounded. It is bounded and periodic with an even period when m is of the form $2\sin\frac{2\ell\pi}{k}$. Also for $-2 < m < 2$ the sequence values lie on two ellipses when plotted in the complex plane. Next step in our research is to find the general equations of the ellipses for any m satisfying $-2 < m < 2$ and analyze the cases where $m > 2$, $m = 2$, and $m < -2$.

References

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[3] Saber N. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag New York, Inc., 1999.