GENERALIZING RANDOM FIBONACCI SEQUENCES

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ABSTRACT. We consider generalized Fibonacci sequences with recurrence relation $x_{n+p+1} = x_{n+p} + x_n$, which have growth rates of the form $\lim_{n\to\infty} |x_n|^{1/n}$ that behave similarly to the golden ratio, $(1+\sqrt{5})/2$. Following Makover and McGowan's analysis of the random Fibonacci sequence, we find bounds for the expectede value, $E(|x_n|)^{1/n}$, for random sequences given by $x_{n+p+1} = \pm x_{n+p} + x_n$. Finally, we further generalize these random sequences using two parameters, p and q, and we experimentally observe how $\lim_{n\to\infty} |x_n|^{1/n}$ contains surprising information about the divisors of $q+1$.

1. Generalized Fibonacci p-Sequences

In order to generalize the famous Fibonacci sequence [4], consider the original problem formulated by Leonardo de Pisa. Suppose that a pair of newborn rabbits are kept in an enclosure, and every month after maturing they reproduce. Visualize a sequence of rabbit pairs, with N representing a pair of newborn rabbits, and M representing a pair of mature rabbits. Newborn pairs mature in a month $(N \mapsto M)$, and mature rabbits reproduce every month $(M \mapsto MN)$.

$$M \to MN \to MNM \to MNMMN \to MNMMNMNM \to \cdots$$

The number of pairs of rabbits in the n^{th} month is the n^{th} Fibonacci number.

Now consider that instead of taking 1 month to mature, a pair of rabbits takes p months to mature. Let $N_p, \ldots, N_1$ denote the pre-maturity stages, with N_k representing a pair of rabbits that will mature in k months. For $p = 2$, we have the following sequence:

$$M \to MN_2 \to MN_2N_1 \to MN_2N_1M \to MN_2N_1MMN_2 \to \cdots$$

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A recurrence relation similar to the Fibonacci sequence can be found for these sequences, given by

$$x_{n+p+1} = x_{n+p} + x_n$$

with $x_n = 1$ for $1 \leq n \leq p+1$. We will call such sequences *generalized Fibonacci p-sequences.*

This recurrence relation is a homogeneous difference equation and has characteristic polynomial $A_p(x) = x^{p+1} - x^p - 1$. For $p = 1$, we have the polynomial whose largest root is the golden ratio, $\phi = (1+\sqrt{5})/2$. We will denote the largest real root of these characteristic polynomials by λ_p. Analogous to the Fibonacci sequence, we have

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_p.$$

Proposition 1.1. *Each λ_p is irrational and in the interval $(1, \phi]$.*

Proof. Since $A_p(x)$ has leading coefficient 1, Cauchy's bound for polynomial roots provides an upper bound of $1 + \max\{|-1|, |-1|\} = 2$. Via Descartes' rule of signs, $A_p(x)$ has one positive root. By the rational root theorem, all possible rational roots of $A_p(x)$ are ± 1. However, neither $A_p(-1)$ nor $A_p(1)$ are zero. Thus, $A_p(x)$ only has irrational positive roots.

Since $A_p(1) = -1 < 0$ and $A_p(\phi) = \phi^{p+1} - \phi^p - 1 = \phi^{p+2} - 1 > 0$, we have $\lambda_p \in (1, \phi)$.

Proposition 1.2. *The sequence $(\lambda_p)_{p=1}^{\infty}$ is strictly decreasing.*

Proof. Consider $p-1$ and p with corresponding $A = \lambda_{p-1}$ and $B = \lambda_p$. Thus,

$$A^p = A^{p-1} + 1$$
$$B^{p+1} = B^p + 1.$$

Dividing by B in the second equation and subtracting yields

$$A^p - B^p = A^{p-1} - B^{p-1} + 1 - \frac{1}{B}.$$

Then, by factoring and dividing by $A - B$, we have

$$\sum_{k=0}^{p-1} A^k B^{p-1-k} = \frac{B-1}{B(A-B)} + \sum_{k=0}^{p-2} A^k B^{p-2-k}$$

$$A^{p-1} + \sum_{k=0}^{p-2} A^k (B^{p-1-k} - B^{p-2-k}) = \frac{B-1}{B(A-B)}.$$

Since $B > 1$, the left side is positive. Thus, $A > B$ and the sequence of λ_p's is strictly decreasing.

Proposition 1.3.

$$\lim_{p\to\infty} \lambda_p = 1$$

We present the proof below, originally proposed by Chelst [1].

Proof. Assume $p \geq 2$. Let $r = (p+1)^{1/p}$. We will show that

$$x_n < r^n.$$

For $n \leq p+1$, we have $x_n < r^n$ since $r > 1$. For $n = p+2$,

$$x_{p+2} = 2 \leq p+1 = r^p < r^{p+2}.$$

For the inductive step, assume that $x_m < r^m$ for all $m \leq n$. Then,

$$\begin{aligned}
x_{n+1} &= x_n + x_{n-p} \\
&< r^n + r^{n-p} \\
&= r^n(1 + r^{-p}) \\
&= r^n\left(1 + \frac{1}{p+1}\right) \\
&= r^n\left[\left(1 + \frac{1}{p+1}\right)^p\right]^{1/p} \\
&< r^n e^{1/p} \\
&< r^n (p+1)^{1/p} \\
&= r^{n+1}.
\end{aligned}$$

Also,

$$\lim_{p\to\infty} (p+1)^{1/p} = 1.$$

Since $1 < \lambda_p < r$,

$$\lim_{p\to\infty} \lambda_p = 1.$$

2. Bounds for Random Fibonacci 2-Sequences

Now consider the collection of sequences with the recurrence relation

$$x_{n+p+1} = \pm x_{n+p} + x_n,$$

with each plus or minus independent and randomly selected with equal probability. We will call these *random Fibonacci p-sequences*. Viswanath [6] has

proven that for $p = 1$, this sequence yields a "golden ratio-like" number, analogous to the λ_p's from Section 1. In particular,

$$\lim_{n\to\infty} \sqrt[n]{|x_n|} = 1.13198824\ldots$$

with probability 1, Furstenberg [3]. We desire to find growth rates for the random Fibonacci p-sequences as well. We denote the value of this limit as $\widetilde{\lambda_p}$.

For this section, we consider $p = 2$. The recurrence relation for this class of random sequences is $x_{n+3} = \pm x_{n+2} + x_n$. We seek to find the value of $\widetilde{\lambda_2}$, which we have experimentally computed as

$$\widetilde{\lambda_2} \approx 1.1267\ldots$$

The convergence of this value can be observed in Figure 1. Just as Makover and McGowan [5] found bounds on the growth rate when $p = 1$, we consider the growth rate of $E(|x_n|)^{1/n}$ for $p = 2$.

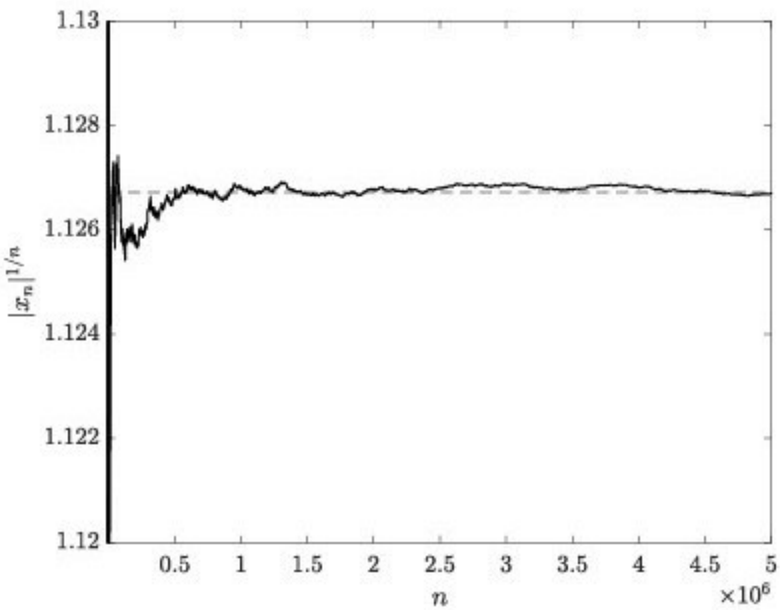

FIGURE 1. A typical random Fibonacci 2-sequence up to $n = 5000000$, demonstrating the almost-sure convergence of $\lim_{n\to\infty} |x_n|^{1/n}$ to $\widetilde{\lambda_2}$.

As in [5] consider the binary tree in Figure 2 of possible sequences corresponding to each choice of plus or minus. We will use this to bound the expected value $E(|x_n|)^{1/n}$. We will similarly consider the recurrence relation $x_{n+3} = |\pm x_{n+2} + x_n|$, which can be seen as merely the reversal of the orientation of the nodes. Notably, this will not affect the sum along a row of the tree for the purposes of exploring $E(|x_n|)$ and will allow us to assume each entry is non-negative.

Consider the subtree starting from a parent a on the $(n-4)^{\text{th}}$ row of the original tree. See Figure 2 for a visualization of this subtree. We will denote the children of a as b_1 and b_2. It is not possible for both children of a to be less than a, so we assume that $b_1 \geq a$. Similarly, we can assume that $c_1 \geq b_1$ and $c_3 \geq b_2$.

We will denote the sum along the n^{th} row of the entire tree of possibilities as $S(n)$, and the sum along the corresponding n^{th} row of the subtree with parent node a as $S^*(n)$. Thus, the values for these row sums can be given by

$$S^*(n-4) = a$$
$$S^*(n-3) = b_1 + b_2$$
$$S^*(n-2) = c_1 + c_2 + c_3 + c_4$$
$$S^*(n-1) = 3a + 2c_1 + c_2 + c_3 + c_4 + |a - c_2| + |a - c_3| + |a - c_4|$$
$$\begin{aligned} S^*(n) = {} & 5a + b_1 + 3b_2 + 3c_1 + c_2 + 2c_3 + c_4 + |a + b_1 - c_1| + |a - b_1 + c_2| \\ & + |a - c_2| + |b_1 - |a - c_2|| + |a - c_3| + |b_2 - |a - c_3|| + |a - b_2 + c_4| \\ & + |a - c_4| + |b_2 - |a - c_4||. \end{aligned}$$

We now seek to bound $S(n)$ by linear combinations of the previous row sums.

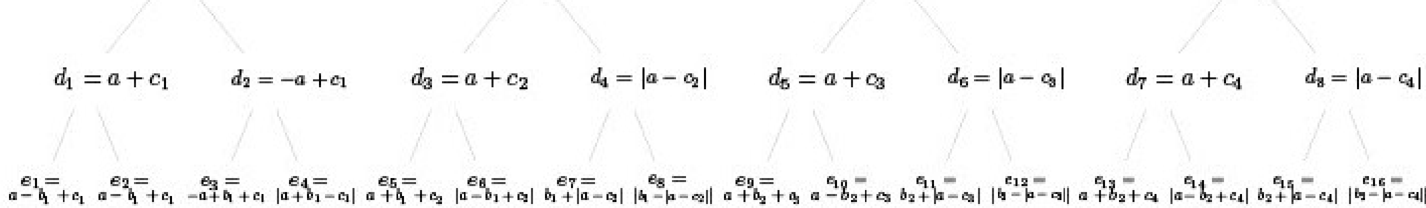

FIGURE 2. A subtree of the tree of possible outcomes of the random Fibonacci 2-sequence, generated by selecting an arbitrary parent node, a. We consider the final row as the n^{th} row.

Lemma 2.1.

$$8S(n-4)+2S(n-3)+2S(n-2) \leq S(n)$$

Proof. For several of the terms with absolute values, we omit the absolute values to create lower bounds:

$$\begin{aligned} S^*(n) &\geq 5a+b_1+3b_2+3c_1+c_2+2c_3+c_4+(a+b_1-c_1)+(a-b_1+c_2) \\ &\quad +|a-c_2|+(b_1-|a-c_2|)+|a-c_3|+(b_2-|a-c_3|)+(a-b_2+c_4) \\ &\quad +|a-c_4|+(b_2-|a-c_4|) \\ &= 8a+2b_1+4b_2+2c_1+2c_2+2c_3+2c_4 \\ &\geq 8S^*(n-4)+2S^*(n-3)+2S^*(n-2). \end{aligned}$$

Naturally, numerous combinations of these absolute value omissions may be chosen, but this one is found to be optimal for our purposes.

Lemma 2.2.

$$S(n) \leq 3S(n-4)+6S(n-3)+2S(n-2)+S(n-1)$$

Proof. We will first find upper bounds for the terms $|a+b_1-c_1|$, $|a-b_1+c_2|$, and $|b_1-|a-c_2||$ under the assumption that $a \leq b_1 \leq c_1$.

A. $|a+b_1-c_1| \leq c_1$

(1) $(a+b_1 \geq c_1)$

$$\begin{aligned} |a+b_1-c_1| &= a+(b_1-c_1) \\ &\leq a \\ &\leq c_1 \end{aligned}$$

(2) $(a+b_1 < c_1)$

$$\begin{aligned} |a+b_1-c_1| &= c_1-a-b_1 \\ &\leq c_1 \end{aligned}$$

B. $|a-b_1+c_2| \leq -a+b_1+c_2$

(1) $(a+c_2 \geq b_1)$

$$\begin{aligned} |a-b_1+c_2| &= c_2+(a-b_1) \\ &\leq c_2 \\ &\leq c_2+(b_1-a) \end{aligned}$$

(2) $(a + c_2 < b_1)$

$$\begin{aligned} |a - b_1 + c_2| &= (b_1 - a) - c_2 \\ &\le b_1 - a \\ &\le c_2 + (b_1 - a) \end{aligned}$$

C. $|b_1 - |a - c_2|| \le -a + b_1 + c_2$

(1) $(a \ge c_2)$

$$\begin{aligned} |b_1 - |a - c_2|| &= |b_1 - (a - c_2)| \\ &= |(b_1 - a) + c_2| \\ &= b_1 - a + c_2 \end{aligned}$$

(2) $(a < c_2)$

$$\begin{aligned} |b_1 - |a - c_2|| &= |b_1 - (c_2 - a)| \\ &\le b_1 + c_2 - a \end{aligned}$$

For several other terms, the use of the triangle inequality gives the best general upper bounds. Hence,

$$\begin{aligned} S^*(n) \le\ & 5a + b_1 + 3b_2 + 3c_1 + c_2 + 2c_3 + c_4 + (c_1) + (-a + b_1 + c_2) + |a - c_2| \\ & + (-a + b_1 + c_2) + |a - c_3| + (a + b_2 + c_3) + (a + b_2 + c_4) + |a - c_4| \\ & + (a + b_2 + c_4) \\ =\ & 6a + 3b_1 + 6b_2 + 4c_1 + 3c_2 + 3c_3 + 3c_4 + |a - c_2| + |a - c_3| + |a - c_4| \\ \le\ & 3S^*(n-4) + 6S^*(n-3) + 2S^*(n-2) + S^*(n-1). \end{aligned}$$

Theorem 2.3.

$$1.07802\ldots \le E(|x_n|)^{1/n} \le 1.35395\ldots$$

Proof. Firstly, $S(n)$ is a sequence bounded by recurrence relations, given by Lemmas 1 and 2. This implies that the growth rate of $S(n)$ can be bounded by the roots of the characteristic polynomials of the bounds, the quartics $x^4 - 2x^2 - 2x - 8$ and $x^4 - x^3 - 2x^2 - 6x - 3$. Then, since $E(|x_n|) = S(n)/2^n$, we can bound the value for $E(|x_n|)^{1/n}$ using the roots of these characteristic polynomials divided by 2.

Jensen's inequality preserves the lower bound for $\widetilde{\lambda_2}$. It is reasonable to expect that these bounds hold for $\widetilde{\lambda_2}$ following Viswanath's success in computing $\widetilde{\lambda_1}$ [6], Makover and McGowan's success in bounding $\widetilde{\lambda_1}$ this way [5], and the approximation for $\widetilde{\lambda_2}$ satisfying these bounds as well.

3. General Bounds for Random Fibonacci p-Sequences

We now consider the case for general p, still using the approach of the tree of possible values for $|x_n|$. We can find looser bounds for any p, which provides a valuable result on the values of $\widetilde{\lambda}_p$.

Within the tree of possible values of $|x_n|$, we again select an arbitrary subtree. The subtree we select has its root as the $n-p-1^{\text{th}}$ term. Fix the first $p+1$ rows of this tree as positive unknowns, using the same method as from Section 2. The first row's entry will be labeled a, and the $n-1^{\text{st}}$ row's entries will be labeled $z_1, z_2, \ldots, z_{2^p}$, each of which is assumed to be positive, as in Section 2. Then consider the next row as the n^{th} row, whose entries we label as $\theta_1, \theta_2, \ldots, \theta_{2^{p+1}}$. So,

$$\begin{aligned} S^*(n-p-1) &= a \\ S^*(n-1) &= \sum_{i=1}^{2^p} z_i \\ S^*(n) &= \sum_{i=1}^{2^{p+1}} \theta_i \\ &= \sum_{i=1}^{2^p} (|z_i + a| + |z_i - a|) \end{aligned}$$

Theorem 3.1. *For any p,*

$$1 \leq E(|x_n|)^{1/n} \leq \lambda_p.$$

Proof. Since a and each z_i are positive,

$$S^*(n) = S^*(n-1) + 2^p S^*(n-p-1) + \sum_{i=1}^{2^p} |z_i - a|$$

For a lower bound,

$$S(n-1) + 2^p S(n-p-1) \leq S(n),$$

and for an upper bound, the triangle inequality provides

$$S(n) \leq 2S(n-1) + 2^{p+1} S(n-p-1).$$

Again, the sequence of row sums must have a growth rate between the largest positive root for each characteristic polynomials of the bounds. Those polynomials are $x^{p+1} - x^p - 2^p$ and $x^{p+1} - 2x^p - 2^{p+1}$. Via Descartes' Rule of Signs, these polynomials only have one positive root each. The first polynomial has

a root of 2. Using the substitution $x = 2y$, the second polynomial becomes a multiple of $y^{p+1} - y^p - 1$, which, interestingly, is the characteristic polynomial of the non-random generalized Fibonacci p-sequences. Thus, since $E(|x_n|)^{1/n} = S(n)^{1/n}/2$, we have

$$1 \leq E(|x_n|)^{1/n} \leq \lambda_p.$$

Corollary 3.2.

$$\lim_{p\to\infty} E(|x_n|)^{1/n} = 1$$

The non-random sequences each have an irrational growth rate that decrease monotonically to 1 as p grows, with each lying in the interval $(1, \phi)$. The reappearance of these same growth values as bounds suggests a similar pattern for the random cases. It is not known if Viswanath's number is irrational, but it seems likely.

Conjecture. *With probability* 1,

$$\lim_{n\to\infty} |x_n|^{1/n} = \widetilde{\lambda}_p$$

converges for each p, *each* $\widetilde{\lambda}_p$ *is irrational, and the sequence of* $\widetilde{\lambda}_p$ *decreases monotonically with*

$$\lim_{p\to\infty} \widetilde{\lambda}_p = 1.$$

Due to the difficulty of using a computer-assisted proof for this problem, we instead examine some interesting numerical results that suggest even more profound results for random generalized Fibonacci p-sequences.

4. Generalized Fibonacci (p, q)-Sequences

We will further generalize these non-random sequences by returning to Leonardo de Pisa's rabbit analogy. Forbes and Sullivan [2] consider that in addition to the p months required to mature, the rabbits must also wait q months before they can reproduce again. The states of these mature rabbits can be denoted as $M_q, \cdots M_0$, with M_k representing a pair of mature rabbits that must wait k months to reproduce. For example, the sequence for $p = 2$ and $q = 1$ is

$$M_1 \to M_0N_2 \to M_1N_1 \to M_0N_2M_1 \to M_1N_1M_0N_2 \to \cdots$$

The sequence for $p = 3$ and $q = 2$ is

$$M_1 \to M_0N_3 \to M_2N_2 \to M_1N_1 \to M_0N_3M_1 \to \cdots$$

The sequences generated this way have recurrence relation

$$x_{n+\max\{p,q\}+1} = x_{n+|p-q|} + x_n,$$

with $x_n = 1$ for $n \leq \max\{p,q\} + 1$. We call these *generalized Fibonacci (p,q)-sequences.*

Note that this relation is symmetric about p and q. By supposing $p > q$ and fixing a value of q, the propositions from Section 1 can all be proven in generality for these new sequences. Thus, we now have Fibonacci-like common ratios, $\lambda_{p,q}$, that are irrational, reside in $(1, \phi)$, and decrease monotonically to 1 as $p \to \infty$ for each fixed q.

We are now prepared to discuss the random case, where

$$x_{n+\max\{p,q\}+1} = \pm x_{n+|p-q|} + x_n$$

with each plus or minus independent and randomly selected with equal probability. We call sequences of this form *random Fibonacci (p,q)-sequences.*

We take an experimental approach, and examine the plots of approximations of $\widetilde{\lambda_{p,q}}$ for each q, with p allowed to vary. These result in highly interesting plots that are worth further efforts interpreting. They are posted at http://ww2.coastal.edu/oarslan/fplots. Figure 3 contains selected plots for $q = 0, 1, 3, 7$.

Note that some points on this plot can be separated into different families of points which are strikingly visible. For example, if $q = 7$, this plot appears to contain 4 of these distinct "curves." It is also notable that the lowest of these spikes, corresponding to the lowest curve, occurs at every 8^{th} p-value. In particular, they occur at the p-values $15, 23, 31, 39, \ldots$

For this example, the p-values that lie on the lowest curve are congruent to 7 mod 8. The p-values of the next lowest curve are congruent to 3 mod 8. The next curve up has p-values 1 less than a multiple of 2 that is neither 1 mod 8 or 5 mod 8. The upper curve has the remainder of the p-values. It appears that the divisors of 8 determine these points.

Similarly, in Figure 3 we also see that $q = 3$ provides 3 curves, one for each divisor of 4. For any prime value of $q + 1$, the corresponding plot appears to only oscillate between two curves, with one having "period" $q + 1$. Finally, the $q = 0$ case appears monotonic, as 1 has only itself as a divisor. In this case, the following conjecture goes in tandem with Conjecture 2.

Conjecture. *For any fixed q, let $d_1, \ldots, d_{\sigma(q+1)}$ be the divisors of $q + 1$. Define the set $P_d = \{p > q \mid \gcd(p+1, q+1) = d\}$. Then $\left\{P_{d_1}, \ldots, P_{d_{\sigma(q+1)}}\right\}$ forms a partition of the naturals greater than q such that the subsequence $\left(\widetilde{\lambda_{p,q}} \mid p \in P_d\right)$ is monotonic for each divisor d of $q + 1$.*

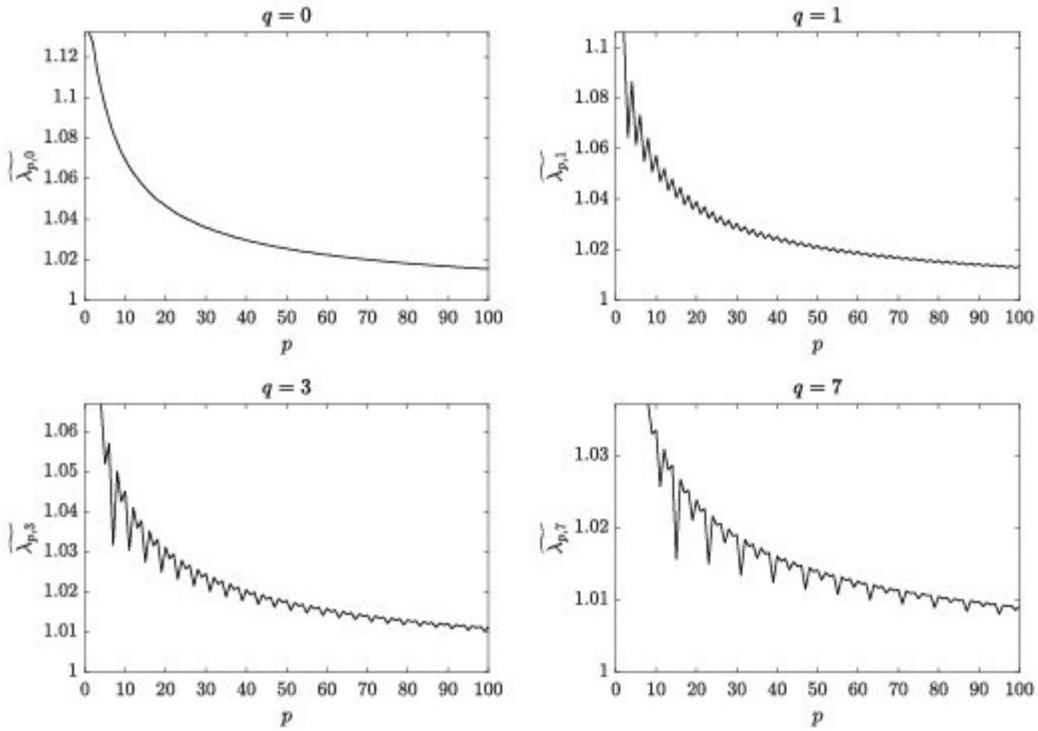

FIGURE 3. Plots of $\widetilde{\lambda_{p,q}}$ for $q = 0, 1, 3, 7$ and varying values of p in the range $q < p \leq 100$. Each selected q-value exhibits different behavior in its apparent oscillations.

Figure 4 serves as an example of these partitions, and the resulting subsequences of $\left(\widetilde{\lambda_{p,q}}\right)_{p=q+1}^{\infty}$ that appear to be monotonic. It also appears that the subsequences corresponding to the divisors 1 and $q+1$ lie on curves that form upper and lower bounds for the sequence.

In conclusion, there is much more to discover about these classes of sequences. The non-random (p, q)-Fibonacci sequences have many properties worth exploring, such as their values modulo k. Approximations of the values via tighter bounds for the random Fibonacci p-sequence growth rates may be fruitful to pursue further. Furthermore, the cause of these distinct curves appearing from the random Fibonacci (p, q)-sequences with such clarity is mysterious. The means by which the growth rates of the generalized

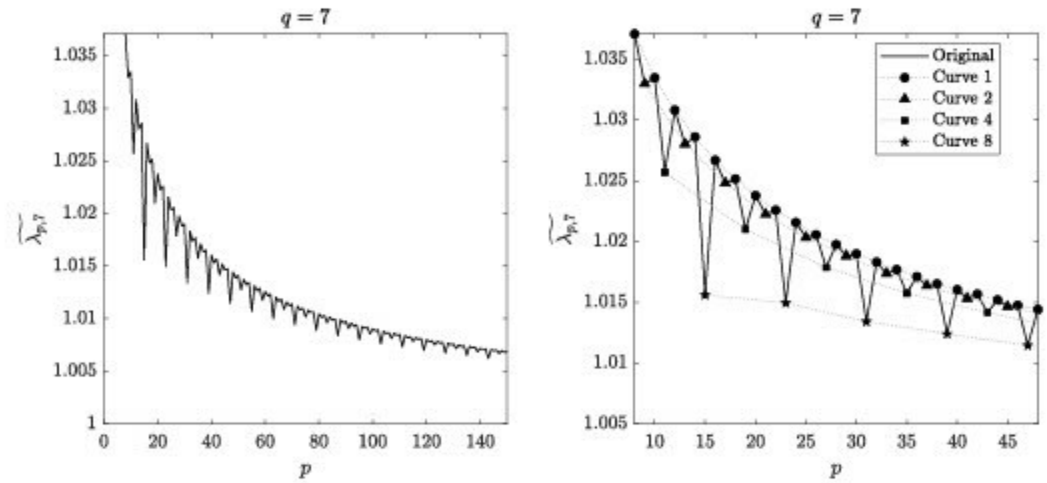

FIGURE 4. Plot of $\widetilde{\lambda_{p,7}}$ for varying values of $8 \leq p \leq 150$ (left), as well as a plot suggesting 4 notable subsequences, one for each divisor of 8 (right).

random Fibonacci sequences contain information about the divisors of of its parameters is a bizarre pattern that begs for further investigation.

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