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A Fixed Point Property for the Lorentz Space $L_{p,1}(\mu)$

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Abstract. Let (X,Σ,μ) be a σ–finite measure space. Our main result is that every non–expansive mapping from a weak–star compact convex subset of $L_{p,1}(\mu)$ into itself has a fixed point. We prove this by showing that $L_{p,1}(\mu)$ has the weak–star uniform Kadec–Klee property.

1. Introduction. Let (X,Σ,μ) be a σ–finite measure space. For a μ–measurable function f we define the distribution of f by $d_f(t) = \mu(\{x : |f(x)| > t\})$, $0 < t < \infty$, and the decreasing rearrangement of f by $f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}$. For $1 < p < \infty$, the Lorentz space $L_{p,1}(\mu)$ is the Banach space of all (equivalence classes of) μ–measurable functions f under the norm $\|f\| = \int_0^\infty f^*(t)\, d(t^{1/p})$. (Throughout, we shall always take $1 < p < \infty$.) $L_{p,1}(\mu)$ is a dual space whose predual is the closure in $L_{p,1}(\mu)^* = L_{p',\infty}(\mu)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) of the μ–integrable simple functions, under the usual pairing $(f,g) = \int_X fg\, d\mu$.

Let C be a closed bounded convex subset of a Banach space $(E, \|\cdot\|)$. A map $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all x, $y \in C$. The main result of the paper is that $L_{p,1}(\mu)$ enjoys the weak–star fixed point property.

Theorem. *Let C be a weak–star compact convex subset of $L_{p,1}(\mu)$. Then, every nonexpansive mapping on C has a fixed point.*

In [4] van Dulst and Sims proved that a dual Banach space with a weak–star sequentially compact unit ball enjoys the weak–star fixed point property, provided that a certain convexity condition is satisfied. To be precise, a dual space E has the *weak–star uniform Kadec-Klee property* if the following holds: given $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that for every weak–star convergent sequence (f_n) with $\|f_n\| = 1$, $\|f_n - f_m\| \geq \varepsilon$ $(m \neq n)$, and with weak–star limit f, we

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have $\|f\| \leq 1-\delta(\varepsilon)$. This property and its counterpart for the weak topology were introduced by Huff [6]. Applied to the Lebesgue spaces $L_p(\mu)$, $1 < p < \infty$, for example, this property amounts to a uniform version of the classical Radon–Riesz Theorem on the coincidence of weak and norm convergence for sequences on the unit sphere of $L_p(\mu)$.

The hard work in the present paper will come in the proof that $L_{p,1}(\mu)$ has the weak–star uniform Kadec–Klee property (Theorem 3.2). Sedaev [10] showed that a general class of Lorentz spaces has the (non–uniform) weak–star Kadec–Klee property: that is, if $f_n \to f$ weak–star and if $\|f_n\| \to \|f\|$, then $\|f - f_n\| \to 0$. But we do not know whether his proof can be modified to yield the uniform version of this property. Our proof utilizes a very natural representation for the elements of the unit sphere of $L_{p,1}[0,\infty)$; this is described in Section 2. The main results are given in Section 3.

Finally, a bit more notation. For a set $A \subset [0,\infty)$, $|A|$ will denote its Lebesgue measure and $I(A)$ will denote its characteristic function. If $0 < |A| < \infty$, we write $e(A) = I(A)/|A|^{1/p}$ (so that $e(A)$ is of norm one in $L_{p,1}[0,\infty)$). And for a measurable function f, $\{f > t\}$ denotes the set $\{s : f(s) > t\}$. We shall also use the fact that $L_{p,1}(\mu)$ satisfies a lower p–estimate for disjoint elements; that is, if f and g are disjointly supported elements in $L_{p,1}(\mu)$, then $\|f+g\|^p \geq \|f\|^p + \|g\|^p$ [3].

2. A Representation Theorem. The proofs of our main results depend heavily on the representation of a given element on the unit sphere of $L_{p,1}[0,\infty)$ as the barycenter of a probability measure supported on the extreme points of the unit ball. It can be shown that this representation is *unique*; this theme is developed for general Lorentz spaces in [2]. For the purpose at hand, however, it suffices to prove the existence of the required representation (Proposition 2.2), from which we deduce the main technical ingredient (Proposition 2.4) in our proof of Theorem 3.2.

Lemma 2.1. *Let f be a nonnegative decreasing function on $(0,\infty)$ with $\|f\| = 1$. There exists a probability measure μ on $(0,\infty)$ such that*

$$f = \int_0^\infty e\big((0,u)\big)\, d\mu(u)$$

(as a Bochner integral).

Proof. We may assume that f is right–continuous, so that $f = f^*$. Let μ be the probability measure defined by $\mu(A) = \int_A u^{1/p} d\big(-f^*(u)\big)$ for every Borel set A. By right–continuity we have $f^*(u) = \int_{(u,\infty)} t^{-1/p}\, d\mu(t)$ for $0 < u < \infty$. It follows that $f = \int_0^\infty e\big((0,u)\big)\, d\mu(u)$. □

Proposition 2.2. *Let f be a nonnegative function on $(0,\infty)$ with $\|f\| = 1$. There exist a collection of Borel sets $\big(A(u)\big)_{u>0}$ and a probability measure μ on $(0,\infty)$ with the following properties:*

(i) *$A(u) \subset A(v)$, except for a set of measure zero, if $u < v$;*
(ii) *$\big|A(u)\big| = |u|$;*
(iii) *$f = \int_0^\infty e\big(A(u)\big)\, d\mu(u)$;*
(iv) *$f^* = \int_0^\infty e\big((0,u)\big)\, d\mu(u)$.*

Proof. We first define the sets $A(u)$. If f^* is not constant in a neighborhood of u, let $A(u) = \big\{f > f^*(u)\big\}$. If $[u_0,u_1)$ is a maximal interval on which f^* takes the value λ, let τ be an isomorphism from the measure algebra of $[u_0,u_1)$ onto the measure algebra of $\{f = \lambda\}$, and define $A(u) = A(u_0) \cup \tau\big((u_0,u)\big)$ for $u_0 < u < u_1$. Plainly, (i) and (ii) are satisfied. Let $S(f) = \big\{e\big(A(u)\big) : u > 0\big\}$ and let $S(f^*) = \big\{e(0,u);\ u > 0\big\}$. Define a mapping T from $S(f^*)$ to $S(f)$ by $T\big(e(0,u)\big) = e\big(A(u)\big)$. By linearity, T extends to a map from the convex hull of $S(f^*)$ onto the convex hull of $S(f)$. From conditions (i) and (ii), it is clear that T is an affine isometry. So T extends to an affine isometry from the closed convex hull of $S(f^*)$ onto the closed convex hull of $S(f)$. Moreover, it is clear from the definition of the sets $\big(A(u)\big)$ that $T(f^*) = f$. By Lemma 2.1, there exists a probability measure μ such that $f^* = \int_0^\infty e\big((0,u)\big)\, d\mu(u)$. Composing with T we see that $f = \int_0^\infty e\big(A(u)\big)\, d\mu(u)$. Thus (iii) and (iv) are satisfied. □

Lemma 2.3. *Let $\varepsilon > 0$. Suppose that $A \subset [0,\infty)$ with $|A| = u > 0$, and that*

$$\int_0^\infty e(A)(t)\, d(t^{1/p}) > 1 - \varepsilon^2.$$

Then $\big|A \setminus [0,u]\big| \le C_1 \varepsilon u$ and, hence, $\|e(A) - e\big((0,u)\big)\|^p \le C_2^p \varepsilon$, where C_1 and C_2 are constants depending only on p.

Proof. Suppose that $\big|A \setminus [0,u]\big| = \alpha u$. Then,

$$\begin{aligned}
\varepsilon^2 &> 1 - \int_0^\infty e(A)(t)\, d(t^{1/p}) \\
&= u^{-1/p} \int_0^u d(t^{1/p}) - u^{-1/p} \int_A d(t^{1/p}) \\
&\ge u^{-1/p} \int_{u(1-\alpha)}^u d(t^{1/p}) - u^{-1/p} \int_u^{u(1+\alpha)} d(t^{1/p}) \\
&= \frac{1}{p}\left(1 - \frac{1}{p}\right)\alpha^2 + O(\alpha^4).
\end{aligned}$$

The result follows at once. □

Proposition 2.4. *Let $\varepsilon > 0$. Suppose that $\|f\| = 1$ while $\int_0^\infty f(t)\,d(t^{1/p}) > 1-\varepsilon^{3p}$. Then, $\|f - f^*\| < C_3\varepsilon$, where C_3 is a constant depending only on p.*

Proof. Let f^+ and f^- denote the positive and negative parts of f. We have

$$\int_0^\infty f^+(t)\,d(t^{1/p}) \geq \int_0^\infty f(t)\,d(t^{1/p}) > 1-\varepsilon^{3p},$$

and so $\|f^+\| > 1-\varepsilon^{3p}$. Since $L_{p,1}$ satisfies a lower p–estimate for disjoint vectors, we obtain $\|f^-\|^p \leq \|f\|^p - \|f^+\|^p \leq 1-(1-\varepsilon^{3p})^p \leq p\varepsilon^{3p}$, whence $\|f - |f|\,\| \leq 2p^{1/p}\varepsilon^3$. We can associate with $|f|$ a Borel probability measure μ and a collection of sets $\big(A(u)\big)_{u>0}$ having the properties described in Proposition 2.2. Then:

$$\begin{aligned}\varepsilon^{3p} &> 1 - \int_0^\infty |f(t)|\,d(t^{1/p}) \\ &= \int_0^\infty \big(f^*(t) - |f(t)|\big)\,d(t^{1/p}) \\ &= \int_0^\infty \left(\int_0^\infty \big(e((0,u)) - e(A(u))\big)\,d\mu(u)\right)(t)\,d(t^{1/p}).\end{aligned}$$

Recall the general fact that if $\int h\,dv$ is a Bochner integral taking values in the Banach space E, then $x^*\left(\int h\,dv\right) = \int x^*(h)\,dv$ for all $x^* \in E^*$. Thus we may interchange the order of integration to obtain $\varepsilon^{3p} > \int_0^\infty g(u)\,d\mu(u)$, where

$$g(u) = \int_0^\infty \big(e((0,u)) - e(A(u))\big)(t)\,d(t^{1/p}).$$

Observe that $0 \leq g(u) \leq 1$, and so $\mu\big(\{u : g(u) \geq \varepsilon^{2p}\}\big) < \varepsilon^p$ by Chebyshev's inequality. It now follows from Lemma 2.3 that $\mu\big(\{u : \|e(A(u)) - e((0,u))\| > C_2\varepsilon\}\big) < \varepsilon^p$. By the triangle inequality,

$$\|\,|f| - f^*\| \leq \int_0^\infty \|e(A(u)) - e((0,u))\|\,d\mu(u) \leq C_2\varepsilon + 2\varepsilon^p.$$

The desired result follows after another application of the triangle inequality. □

3. Main Results. In this section we prove that $L_{p,1}(\mu)$ has the weak–star fixed point property. We begin with the technical Lemma 3.1, then prove Theorem 3.2 (that $L_{p,1}(\mu)$ has the weak–star uniform Kadec–Klee property). By using known results from [1], [4], and [7], we then show that $L_{p,1}(\mu)$ has the weak–star fixed point property.

Lemma 3.1. *Let f be a nonnegative function on $(0,\infty)$ with $\|f\| = 1$. Given $\varepsilon > 0$, there exists a surjective isometry T of $L_{p,1}[0,\infty)$, which is also a weak–star automorphism, such that $\|T(f) - f^*\| < \varepsilon$.*

Proof. Let τ be any measure algebra automorphism of $(0,\infty)$. Then τ induces, in a natural way, a surjective isometry T on $L_{p,1}[0,\infty)$ which is also a weak–star automorphism. By approximating f by a simple function and applying Carathéodory's Theorem (e.g., [9, p. 399]), the construction of an automorphism τ of $(0,\infty)$ yielding the required T is straightforward. □

Theorem 3.2. *Let (X,Σ,μ) be a σ–finite measure space. Then, $L_{p,1}(\mu)$ has the weak–star uniform Kadec–Klee property.*

Proof. We first prove the result for $X = (0,\infty)$ with Lebesgue measure. Let $\varepsilon > 0$ be given. Suppose that $\|f_n\| = 1$ for all n, that $\|f_n - f_m\| \geq \varepsilon$ $(m \neq n)$, and that (f_n) converges weak–star to f. We may assume that $\|f\| = 1 - \delta$, and we shall show that $\delta \geq \delta(\varepsilon) > 0$. (Note that Sedaev's result [10] implies that $\delta > 0$.) In an effort to make this calculation transparent, we have broken it into several manageable steps. While the quantities δ_1, δ_2, etc. that arise could, in principle, be calculated, we shall settle for the observation that they depend only on the quantity δ introduced above (and on p), and that each tends to zero as δ tends to zero.

1° By applying an isometry of $L_{p,1}$ of the form $g \mapsto \tau g$, where τ is ± 1–valued, we may assume that $f \geq 0$. By Lemma 3.1 we may now assume, by applying a suitable weak–star automorphism–cum–isometry, that $\|f - f^*\| < \delta$.

2°
$$\begin{aligned} \int_0^\infty f(t)\,d(t^{1/p}) &\geq \int_0^\infty f^*(t)\,d(t^{1/p}) - \|f - f^*\| \\ &= \|f\| - \|f - f^*\| \\ &\geq 1 - 2\delta . \end{aligned}$$

3° Choose $0 < m,\ M < \infty$, such that $\int_m^M f(t)\,d(t^{1/p}) \geq 1 - 3\delta$. Since $f_n \to f$ weak–star, we may pass to a subsequence and suppose

$$\int_m^M f_n(t)\,d(t^{1/p}) \geq 1 - 4\delta$$

for all n. Since $\|f_n\| = 1$, the lower p–estimate implies

$$\left\| f_n I\big((0,m)\big) + f_n I\big((M,\infty)\big) \right\| \leq \delta_1 .$$

Thus, $\int_0^\infty f_n(t)\,d(t^{1/p}) \geq 1 - 4\delta - \delta_1$ for all n.

4° By Proposition 2.4, we have $\|f_n - f_n^*\| \le \delta_2$.

5° By Helly's Selection Theorem we may assume, by passing to a subsequence, that $f_n^* \to g$ pointwise (almost everywhere) and, in particular, that $f_n^* \to g$ weak–star. Since $f_n^* - f_n \to g - f$ weak–star, we have $\|g - f\| \le \liminf_{n\to\infty} \|f_n^* - f_n\| \le \delta_2$. Hence $\|g\| \ge \|f\| - \delta_2 = 1 - \delta_3$.

6° Select $0 < m_1,\ M_1 < \infty$ such that $\|gI((m_1,M_1))\| \ge 1 - 2\delta_3$. By Egorov's Theorem, $\|f_n^* I((m_1,M_1)) - gI((m_1,M_1))\| \to 0$, so by passing to a subsequence we may suppose that $\|f_n^* I((m_1,M_1)) - gI((m_1,M_1))\| \le \delta_3$ for all n. In particular, we get $\|f_n^* I((m_1,M_1))\| \ge 1 - 3\delta_3$.

7° Since $\|f_n\| = 1$ and $\|g\| \le 1$, it follows from the lower p–estimate and step 6° that $\|f_n^* - f_n^* I((m_1,M_1))\| \le \delta_4$ and $\|g - gI((m_1,M_1))\| \le \delta_4$. Consequently,

$$\begin{aligned}\|f_n^* - g\| &\le \|f_n^* - f_n^* I((m_1,M_1))\| + \|f_n^* I((m_1,M_1)) - gI((m_1,M_1))\| \\ &\qquad + \|gI((m_1,M_1)) - g\| \\ &\le \delta_4 + \delta_3 + \delta_4 = \delta_5 .\end{aligned}$$

8° For $m \ne n$, $\|f_n^* - f_m^*\| \le 2\delta_5$.

9° Finally, combining steps 4 and 8 (and the hypothesis $\|f_n - f_m\| \ge \varepsilon$ $(m \ne n)$), we have $\varepsilon \le \|f_n - f_m\| \le \|f_n - f_n^*\| + \|f_n^* - f_m^*\| + \|f_m^* - f_m\| \le 2\delta_2 + 2\delta_5$. Since $2\delta_2 + 2\delta_5 \ge \varepsilon$, it follows that there is a $\delta(\varepsilon) > 0$ such that $\delta \ge \delta(\varepsilon)$. This completes the proof for $L_{p,1}[0,\infty)$. □

The general case. Now suppose that (f_n) and f are in $L_{p,1}(\mu)$, where (X,Σ,μ) is a σ–finite measure space. Let Σ' be a separable σ–finite sub–σ–algebra for which (f_n) and f are measurable. If Σ' has no atoms, then by Carathéodory's theorem the measure algebra associated with Σ' is isomorphic to the measure algebra of $(0,M)$ for some $0 < M \le \infty$. This isomorphism induces a surjective isometry from $L_{p,1}(0,M)$ onto $L_{p,1}(X,\Sigma',\mu)$ which is also a weak–star automorphism. Now $f_n \to f$ weak–star in $L_{p,1}(X,\Sigma',\mu)$ a fortiori, and so the conclusion follows from the case of $L_{p,1}[0,\infty)$. A slightly more involved but straightforward argument also works for the case where Σ' has atoms. □

Before deducing our main result, we recall the concept of a normal structure. Let E be a Banach space and let C be a closed bounded convex subset of E with at least two points. For $x \in C$, define:

$$\begin{aligned}\mathrm{rad}(x,C) &= \sup\{\|x-y\| \ : y \in C\}, \\ \mathrm{rad}(C) &= \inf\{\mathrm{rad}(x,C) : x \in C\}, \quad \text{and} \\ \mathrm{diam}(C) &= \sup\{\|x-y\| \ : x,y \in C\}.\end{aligned}$$

The set C is said to be *diametral* if $\mathrm{rad}(C) = \mathrm{diam}(C)$. The space E is said to have *normal structure* if E contains no such diametral sets. If E is a dual space, E has *weak–star normal structure* if E contains no weak–star compact convex diametral sets.

Proposition 3.3.
(i) *$L_{p,1}[0,\infty)$ does not have normal structure.*
(ii) *There exists a closed bounded convex subset C of $L_{p,1}[0,\infty)$ and a nonexpansive mapping $T : C \to C$ without a fixed point.*

Proof.
(i) Let C be the closed convex hull of $\{e((0,n)) : n = 1,2,\ldots\}$. Then, C is affinely isometric to the set $C' = \{(a_n) : a_n \geq 0, \sum_n a_n = 1\}$ in ℓ_1. Clearly, C' (and hence C) is diametral.
(ii) The mapping $(a_1,a_2,\ldots) \mapsto (0,a_1,a_2,\ldots)$ is an isometry from C' into C' without a fixed point. □

Theorem 3.4. *Let (X,Σ,μ) be a σ-finite measure space. Then, $L_{p,1}(\mu)$ has a weak–star normal structure. In particular, every nonexpansive mapping from a weak–star compact convex subset of $L_{p,1}(\mu)$ into itself has a fixed point.*

Proof. It is a result of van Dulst and Sims [4] that in every dual space with a weak–star sequentially compact unit ball, the weak–star uniform Kadec–Klee property implies that the space has weak–star normal structure. The fixed point property is a well–known consequence of weak–star normal structure [1, 7], so the theorem will follow once it is established that $L_{p,1}(\mu)$ has a weak–star sequentially compact unit ball. This can be seen in many ways; for example, by a result of Hagler and Johnson [5] every dual space with the Radon–Nikodym property has a weak–star sequentially compact ball, and so it suffices to check that the separable subspaces of $L_{p,1}(\mu)$ have the Radon–Nikodym property. But a separable subspace of $L_{p,1}(\mu)$ is isometric to a subspace of $L_{p,1}[0,\infty)$, a separable dual space, and so has the Radon–Nikodym property. This completes the proof. □

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