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ON THE EQUIVALENCE OF CERTAIN CONSEQUENCES OF THE PROPER FORCING AXIOM

PETER NYIKOS AND LESZEK PIĄTKIEWICZ

Abstract. We prove that a number of axioms, each a consequence of PFA (the Proper Forcing Axiom) are equivalent. In particular we show that TOP (the Thinning-out Principle as introduced by Baumgartner in the *Handbook of set-theoretic topology*), is equivalent to the following statement: If I is an ideal on ω_1 with ω_1 generators, then there exists an uncountable $X \subseteq \omega_1$, such that either $[X]^\omega \cap I = \emptyset$ or $[X]^\omega \subseteq I$.

§1. Introduction. In this paper we study relations between some consequences of the Proper Forcing Axiom (PFA). Among them we consider the Thinning-out Principle (TOP) introduced by Baumgartner in [B], and the partition calculus axiom $\omega_1 \longrightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ proposed by Todorčević in [T]. We show that each of these two axioms can be restated in a simpler way, and then we easily deduce that Todorčević's axiom (which we call Axiom S in this paper) is a consequence of TOP. We will then show how our versions of these axioms give simplified proofs of the applications of these axioms in [B] and [T].

We will show that the following axiom is equivalent to TOP :

AXIOM 0. Let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ be a collection of (countable) subsets of ω_1 , such that for every uncountable $X \subseteq \omega_1$, there exists a countable set $Q \subseteq X$ which cannot be covered by a finite subfamily of \mathcal{S} . Then there exists an uncountable subset of ω_1 which meets each S_α in a finite set.

We will show that the following (weaker) version of Axiom 0 is equivalent to Axiom S :

AXIOM 1. Let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ be a collection of (countable) subsets of ω_1 , such that for every uncountable $X \subseteq \omega_1$, there exists a countable set $Q \subseteq X$ which cannot be covered by a finite subfamily of \mathcal{S} . Then there exists uncountable set $X \subseteq \omega_1$, such that for each $\alpha \in X$, $S_\alpha \cap X$ is a finite set.

Note that the assumption that each S_α is countable is not essential. Any family \mathcal{S} which satisfies the condition in Axioms 0 and 1, clearly consists of countable sets only.

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§2. Definitions and terminology.

2.1. A *partially ordered set* (shortly a *p.o. set*) is a pair $\langle \mathbf{P}, \leq_{\mathbf{P}} \rangle$, such that \mathbf{P} is a nonempty set and $\leq_{\mathbf{P}}$ is a transitive, reflexive and antisymmetric relation on \mathbf{P} . We will sometimes abuse notation and write \mathbf{P} instead of $\langle \mathbf{P}, \leq_{\mathbf{P}} \rangle$. Let $\langle \mathbf{P}, \leq_{\mathbf{P}} \rangle$ be a p.o. set. \mathbf{P} is *proper*, if forcing along \mathbf{P} preserves stationary sets. A subset C of \mathbf{P} is \downarrow -*directed*, if every finite subset of C has a lower bound in C . C is \downarrow -*centered* if every finite subset of C has a lower bound in \mathbf{P} . A subset D of \mathbf{P} is \downarrow -*dense* if for each $p \in \mathbf{P}$, there exists $d \in D$ such that $d \leq_{\mathbf{P}} p$. Given a family \mathcal{D} of \downarrow -dense subsets of \mathbf{P} , a set G is \mathbf{P} -*generic* for \mathcal{D} if it meets each $D \in \mathcal{D}$, is \downarrow -centered, and has the property that if $p \in G$ and $p \leq_{\mathbf{P}} q$ then $q \in G$. \mathbf{P} has *precaliber* \aleph_1 *downwards* if for every uncountable $Q \subseteq \mathbf{P}$, there is an uncountable $R \subseteq Q$ such that R is \downarrow -centered.

The Proper Forcing Axiom is the following statement :

2.2. PFA. If $\langle \mathbf{P}, \leq_{\mathbf{P}} \rangle$ is a proper p.o. set, and $\mathcal{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of \downarrow -dense subsets of \mathbf{P} , then there exists a \mathbf{P} -generic set for \mathcal{D} .

Even though PFA is a very powerful axiom with an elegant statement, it cannot be proven to be consistent with ZFC, as it implies the consistency of large cardinals. This is one of the reasons why different consequences of PFA were introduced and studied. In this paper we concentrate our attention on two of them: the Thinning-out Principle and $\omega_1 \longrightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$.

2.3. The Thinning-out Principle (TOP). Suppose $A, B \subseteq \omega_1$ are uncountable and $\langle T_\alpha : \alpha \in B \rangle$ is such that $T_\alpha \subseteq \alpha$ for all α . Suppose also that for any uncountable $X \subseteq A$, there exists $\beta < \omega_1$ such that $\{X\} \cup \{T_\alpha : \alpha \in B, \alpha > \beta\}$ has the finite intersection property (f.i.p.). Then there exists an uncountable $X \subseteq A$ such that $\forall \alpha \in B, (X \cap \alpha) \setminus T_\alpha$ is finite. Hence in particular there are uncountable $X \subseteq A$ and $Y \subseteq B$ such that $\forall \alpha \in Y, X \cap \alpha \subseteq T_\alpha$.

We will denote by $\text{TOP}(\omega_1)$ a statement of TOP, with the additional assumption that $A = B = \omega_1$; that is, the following axiom:

2.4. $\text{TOP}(\omega_1)$. Suppose $\langle T_\alpha : \alpha \in \omega_1 \rangle$ is such that $T_\alpha \subseteq \alpha$ for all α . Suppose also that for any uncountable $X \subseteq \omega_1$ there exists $\beta < \omega_1$, such that $\{X\} \cup \{T_\alpha : \alpha > \beta\}$ has the f.i.p.. Then there exists an uncountable $X \subseteq \omega_1$ such that $\forall \alpha \in \omega_1, (X \cap \alpha) \setminus T_\alpha$ is finite.

2.5. Graph. Let A be a set. A *graph* on A is any symmetric, irreflexive relation on A . If \mathbf{G} is a graph on A and $I \subseteq A$, then I is \mathbf{G} -*independent* if $I^2 \cap \mathbf{G} = \emptyset$.

2.6. $\omega_1 \longrightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ (Axiom S). If \mathbf{G} is a graph on ω_1 , then either \mathbf{G} has an uncountable independent set, or else there is a pair S, \mathcal{B} such that S is an uncountable subset of ω_1 and \mathcal{B} is an uncountable disjoint family of finite subsets of ω_1 , such that whenever $s \in S$ and $b \in \mathcal{B}$ satisfy $s < \min b$, there is an edge in \mathbf{G} from s to b , more formally, $(\{s\} \times b) \cap \mathbf{G} \neq \emptyset$.

The following is a special case of a Sparse Graph Axiom (see [N]).

2.7. Countably Sparse Graph on ω_1 . A graph \mathbf{G} on ω_1 is *countably sparse*, if for all uncountable $X \subseteq \omega_1$, there exist countable sets $Q \subseteq X$ and $H \subseteq \omega_1$, such that for all finite sets $b \subseteq (\omega_1 \setminus H)$, there exists $\gamma \in Q$ such that $(\{\gamma\} \times b) \cap \mathbf{G} = \emptyset$.

2.8. Countably Sparse Graph Axiom. Every countably sparse graph on ω_1 has an uncountable independent set.

§3. The equivalence of axioms. We begin with a useful technical lemma.

LEMMA 3.1. Let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ be a collection of countable subsets of ω_1 . The following two conditions are equivalent:

- (a) For every uncountable $X \subseteq \omega_1$, there exists a countable $Q \subseteq X$ which cannot be covered by a finite subfamily of \mathcal{S} .
- (b) For every uncountable $X \subseteq \omega_1$, there exist countable sets $Q \subseteq X$ and $H \subseteq \omega_1$, such that Q cannot be covered by a finite subfamily of $\{S_\alpha : \alpha \notin H\}$.

PROOF OF LEMMA 3.1. Clearly (a) implies (b). Now assume that (b) holds. For an uncountable set $X \subseteq \omega_1$ we will say that a pair of countable sets $Q \subseteq X$ and $H \subseteq \omega_1$ “works” for X , if Q cannot be covered by a finite subfamily of $\{S_\alpha : \alpha \notin H\}$.

Fix an uncountable set $X \subseteq \omega_1$. We will show that there exists a countable $Q \subseteq X$ which cannot be covered by a finite subfamily of \mathcal{S} . Define by induction sequences $\langle X_n \rangle_{n \in \omega}$ of uncountable subsets of ω_1 , and $\langle \beta_n \rangle_{n \in \omega}$ of countable ordinals as follows: let $X_0 = X$ and let

$$\beta_0 = \min \{ \gamma \in \omega_1 : \text{the pair } (X_0 \cap \gamma) \subseteq X_0, \gamma \subseteq \omega_1 \text{ “works” for } X_0 \};$$

if X_n and β_n have been defined let

$$(1) \quad X_{n+1} = X_n \setminus \left(\beta_n \cup \bigcup_{\alpha \in \beta_n} S_\alpha \right),$$

and let

$$(2) \quad \beta_{n+1} = \min \{ \gamma \in \omega_1 : \text{the pair } (X_{n+1} \cap \gamma) \subseteq X_{n+1}, \gamma \subseteq \omega_1 \text{ “works” for } X_{n+1} \}.$$

By (b) the sequence $\langle \beta_n \rangle_{n \in \omega}$ is well defined. Also, since all the sets in \mathcal{S} are countable and each β_n is a countable ordinal, each X_n is an uncountable subset of ω_1 . Notice that the sequence $\langle \beta_n \rangle_{n \in \omega}$ is strictly ascending, and that the sequence $\langle X_n \rangle_{n \in \omega}$ is strictly descending. Put $\beta_\omega = \bigcup_{n \in \omega} \beta_n$ and let $Q = X \cap \beta_\omega$.

Claim 3.1.1. Q cannot be covered by a finite subfamily of \mathcal{S} .

Proof of Claim 3.1.1. Assume that for some finite set $\{\alpha_1, \dots, \alpha_n\} \subseteq \omega_1$ we have

$$(3) \quad Q \subseteq S_{\alpha_1} \cup \dots \cup S_{\alpha_n}.$$

We can assume without loss of generality that, for some $1 < k < n$,

$$(4) \quad \alpha_1 < \dots < \alpha_{k-1} < \alpha_k = \beta_\omega < \alpha_{k+1} < \dots < \alpha_n.$$

Since $\beta_n \nearrow \beta_\omega$, there exists an index $m \in \omega$ such that $\alpha_{k-1} < \beta_m$. Now $Q = X \cap \beta_\omega \supseteq X_{m+1} \cap \beta_{m+1}$, hence by (3) $X_{m+1} \cap \beta_{m+1} \subseteq S_{\alpha_1} \cup \dots \cup S_{\alpha_n}$. By (1) we get $X_{m+1} \cap \bigcup \{S_\alpha : \alpha \in \beta_m\} = \emptyset$, and since $\{\alpha_1, \dots, \alpha_{k-1}\} \subseteq \beta_m$ we get

$$(5) \quad X_{m+1} \cap \beta_{m+1} \subseteq S_{\alpha_k} \cup \dots \cup S_{\alpha_n}.$$

But since $\{\alpha_k, \dots, \alpha_n\}$ is disjoint from β_{m+1} , (5) clearly contradicts the fact that the pair $X_{m+1} \cap \beta_{m+1} \subseteq X_{m+1}$, $\beta_{m+1} \subseteq \omega_1$ “works” for X_{m+1} (see (2)). \square

This concludes the proof of the lemma. \square

LEMMA 3.2. *TOP is equivalent to $TOP(\omega_1)$.*

PROOF OF LEMMA 3.2. It is clear that TOP implies $TOP(\omega_1)$. Now assume $TOP(\omega_1)$, and suppose that $A, B \subseteq \omega_1$ and $\langle T_\alpha : \alpha \in B \rangle$ satisfy the hypothesis of TOP. We can assume without loss of generality that $\forall \alpha \in B, T_\alpha \subseteq A$. Let $\alpha : \omega_1 \rightarrow A$ and $\beta : \omega_1 \rightarrow B$ be order preserving bijections. Let $\gamma : \omega_1 \rightarrow \omega_1$ be an order preserving function such that

$$(1) \quad \forall \xi \in \omega_1, \beta(\xi) < \alpha(\gamma(\xi)).$$

Since A is unbounded in ω_1 it is clear that one can find γ as above. For example it can be defined inductively by: $\gamma(\xi) = \min \{ \theta \in \omega_1 : \beta(\xi) < \alpha(\theta) \}$. For each $\psi \in \omega_1$, let

$$(2) \quad T_\psi^\# = \begin{cases} \alpha^{-1}(T_{\beta(\xi)}) & \text{if } \psi = \gamma(\xi), \\ \psi & \text{if } \psi \notin \gamma^{-1}(\omega_1). \end{cases}$$

Claim 3.2.1. $\langle T_\psi^\# : \psi \in \omega_1 \rangle$ satisfies the hypothesis of $TOP(\omega_1)$.

Proof of the claim. First observe that

(3a) if $\psi \notin \gamma^{-1}(\omega_1)$ then $T_\psi^\# = \psi \subseteq \psi$, and

(3b) if $\psi = \gamma(\xi)$ for some $\xi \in \omega_1$, then $T_{\beta(\xi)} \subseteq \beta(\xi) \subseteq \alpha(\gamma(\xi))$ (see (1));

hence by (2) $T_\psi^\# = \alpha^{-1}(T_{\beta(\xi)}) \subseteq \gamma(\xi) = \psi$.

By (3a) and (3b) we get $\forall \psi \in \omega_1, T_\psi^\# \subseteq \psi$. Now let $X \subseteq \omega_1$ be uncountable. $\alpha^{-1}(X)$ is an uncountable subset of A, hence by our assumption, there exists $\beta_0 < \omega_1$ such that $\{ \alpha^{-1}(X) \} \cup \{ T_\alpha : \alpha \in B \text{ and } \alpha > \beta_0 \}$ has the f.i.p.. Choose $\xi_0 \in \omega_1$ so that $\beta(\xi_0) > \beta_0$. We have $T_{\beta(\xi_0)} \in \{ T_\alpha : \alpha \in B \text{ and } \alpha > \beta_0 \}$ and $T_{\beta(\xi_0)} \subseteq \beta(\xi_0) \subseteq \alpha(\gamma(\xi_0))$, hence

$$(4) \quad \{ \alpha^{-1}(X) \cap \alpha(\gamma(\xi_0)) \} \cup \{ T_\alpha : \alpha \in B \text{ and } \alpha > \alpha(\gamma(\xi_0)) \} \text{ has the f.i.p..}$$

We will be done if we can show that

$$(5) \quad \{ X \} \cup \{ T_\psi^\# : \psi > \gamma(\xi_0) \} \text{ has the f.i.p..}$$

Let $\{ \psi_1, \dots, \psi_n \}$ be a finite subset of $\{ \psi : \psi > \gamma(\xi_0) \}$. Without loss of generality we can assume that for some $k \leq n$, we have $\forall i \leq k, \psi_i = \gamma(\xi_i)$ for some $\xi_i > \xi_0$ and $\forall i > k, \psi_i \notin \gamma^{-1}(\omega_1)$. We have

$$(6) \quad X \cap \bigcap_{i=1}^n T_{\psi_i}^\# \supseteq X \cap \bigcap_{i=1}^k T_{\psi_i}^\# \cap \gamma(\xi_0) = X \cap \alpha^{-1} \left(\bigcap_{i=1}^k T_{\beta(\xi_i)} \right) \cap \gamma(\xi_0).$$

Let us look at the image of the last set under α .

$$(7) \quad \begin{aligned} \alpha^{-1} \left(X \cap \alpha^{-1} \left(\bigcap_{i=1}^k T_{\beta(\xi_i)} \right) \cap \gamma(\xi_0) \right) &= \alpha^{-1}(X) \cap \bigcap_{i=1}^k T_{\beta(\xi_i)} \cap \alpha^{-1}(\gamma(\xi_0)) \\ &= \alpha^{-1}(X) \cap \alpha(\gamma(\xi_0)) \cap \bigcap_{i=1}^k T_{\beta(\xi_i)}. \end{aligned}$$

The first equality in (7) holds because α is a bijection, the second because α

is an order preserving bijection. By (4) the last set in (7) is nonempty, hence $X \cap \bigcap_{i=1}^n T_{\psi_i}^\# \neq \emptyset$, and (5) holds. \square

By the claim and $\text{TOP}(\omega_1)$, there is an uncountable $X \subseteq \omega_1$ such that $(X \cap \alpha) \setminus T_\alpha^\#$ is finite for all $\alpha \in \omega_1$. In particular $(X \cap \gamma(\xi)) \setminus \alpha^{-1}(T_{\beta(\xi)})$ is finite for all $\xi \in \omega_1$ (see (2)). Taking the α image of both sides, we conclude that $(\alpha^{-1}(X) \cap \alpha(\gamma(\xi))) \setminus T_{\beta(\xi)}$ is finite for all $\xi \in \omega_1$. Finally since β is a function onto B , (1) implies that

$$\forall \varphi = \beta(\xi) \in B, (\alpha^{-1}(X) \cap \varphi) \setminus T_\varphi \text{ is finite.} \quad \square$$

We are ready now to prove the equivalence of TOP and Axiom 0.

THEOREM 3.3. *TOP is equivalent to Axiom 0.*

PROOF OF THEOREM 3.3.

LEMMA 3.3.1. *TOP implies Axiom 0.*

PROOF OF LEMMA 3.3.1. Assume TOP , and let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ be a collection of subsets of ω_1 , such that for every uncountable $X \subseteq \omega_1$, there exists a countable set $Q \subseteq X$ which cannot be covered by a finite subfamily of \mathcal{S} . Define by transfinite induction a sequence $\langle T(\alpha) \rangle_{\alpha < \omega_1}$ of countable ordinals as follows: let $T(0) = \min\{\gamma \in \omega_1 : S_0 \subseteq \gamma\}$, and if $\langle T(\alpha) \rangle_{\alpha < \beta}$ have been defined let

$$(1) \quad T(\beta) = \min\{\gamma \in \omega_1 : S_\beta \cup \{T(\alpha) : \alpha < \beta\} \subseteq \gamma\}.$$

Put $A = \omega_1$, and let $B = \{T(\alpha) : \alpha \in \omega_1\}$. For each $\gamma = T(\alpha) \in B$, let $T_\gamma = \gamma \setminus S_\alpha$.

Claim 3.3.2. A, B and $\{T_\gamma : \gamma \in B\}$ satisfy the hypothesis of TOP .

Proof of Claim 3.3.2. Clearly B is an uncountable subset of ω_1 , and $\forall \gamma \in B, T_\gamma \subseteq \gamma$. Let X be an uncountable subset of ω_1 . Since \mathcal{S} satisfies the hypothesis of Axiom 0, we can choose a countable subset Q of X which cannot be covered by a finite subfamily of \mathcal{S} . Let $\beta = \sup(Q) + 1$. We will show that $\{X\} \cup \{T_\gamma : \gamma \in B, \gamma > T(\beta)\}$ has the f.i.p.. Let $\gamma_1 = T(\alpha_1), \dots, \gamma_n = T(\alpha_n) \in B$ be such that

$$(2) \quad \forall i \leq n, \gamma_i > T(\beta).$$

Q is not covered by $S_{\alpha_1} \cup \dots \cup S_{\alpha_n}$, hence $Q \setminus (S_{\alpha_1} \cup \dots \cup S_{\alpha_n}) \neq \emptyset$. Also $Q \subseteq \beta$, hence

$$Q \cap (\beta \setminus S_{\alpha_1}) \cap \dots \cap (\beta \setminus S_{\alpha_n}) \neq \emptyset.$$

Since each $\alpha_i > \beta$ (see (1) and (2)), and $Q \subseteq X$, we get

$$X \cap T_{\gamma_1} \cap \dots \cap T_{\gamma_n} \neq \emptyset. \quad \square$$

By the claim and TOP , there is an uncountable $X \subseteq \omega_1$, such that $X \cap (\gamma \setminus T_\gamma)$ is finite for all $\gamma = T(\alpha) \in B$. Since $S_\alpha = T(\alpha) \setminus T_{T(\alpha)}$ for all $\alpha \in \omega_1$, we get that $X \cap S_\alpha$ is finite for all $\alpha \in \omega_1$. This completes the proof of 3.3.1. \square

LEMMA 3.3.3. *Axiom 0 implies TOP.*

PROOF OF LEMMA 3.3.3. Assume Axiom 0. By Lemma 3.2 it is sufficient to show $\text{TOP}(\omega_1)$. Let $\langle T_\alpha : \alpha \in \omega_1 \rangle$ be a collection of countable sets satisfying the hypothesis of $\text{TOP}(\omega_1)$. For each $\alpha \in \omega_1$ put $S_\alpha = \alpha \setminus T_\alpha$.

Claim 3.3.4. $\mathcal{S} = \{S_\alpha : \alpha \in \omega_1\}$ satisfies the hypothesis of Axiom 0.

Proof of Claim 3.3.4. We show that $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ satisfies (b) in Lemma 3.1. Let X be an uncountable subset of ω_1 . Choose $\beta \in \omega_1$ so that $\{X\} \cup \{T_\alpha : \alpha > \beta\}$ has the f.i.p.. Let $Q = X \cap (\beta + 1)$ and let $H = \beta$. We will show that Q and H “work” for X . Let $\{\alpha_1, \dots, \alpha_n\}$ be a finite set of countable ordinals, disjoint from H . Since $\beta + 1 > \beta$ and for each $i \leq n$, $\alpha_i > \beta$, we have $X \cap T_{\beta+1} \cap T_{\alpha_1} \cap \dots \cap T_{\alpha_n} \neq \emptyset$, and since each $T_{\alpha_i} = \alpha_i \setminus S_{\alpha_i}$, where $\alpha_i > \beta$ and $T_{\beta+1} \subseteq \beta + 1$, we get

$$(X \cap (\beta + 1)) \setminus (S_{\alpha_1} \cup \dots \cup S_{\alpha_n}) \neq \emptyset. \quad \square$$

By the claim and Axiom 0, there is an uncountable set $X \subseteq \omega_1$ such that $X \cap S_\alpha$ is finite for all $\alpha \in \omega_1$. Hence $(X \cap \alpha) \setminus T_\alpha$ is finite for all $\alpha \in \omega_1$. \square

This completes the proof of the theorem. \square

Recall that for every set X , $[X]^\omega$ denotes the set of all countable infinite subsets of X . Since Axiom 0 is clearly equivalent to the statement given in the abstract, Theorem 3.3 gives the following new statement of TOP.

3.4. TOP (restated). If I is an ideal on ω_1 with ω_1 generators, such that there is no uncountable $X \subseteq \omega_1$ for which $[X]^\omega \subseteq I$, then there exists an uncountable $X \subseteq \omega_1$, such that $[X]^\omega \cap I = \emptyset$. Hence also for each uncountable $X \subseteq \omega_1$, there exists an uncountable $Y \subseteq X$, such that $[Y]^\omega \cap I = \emptyset$. \square

LEMMA 3.5. *Let \mathbf{G} be a graph on ω_1 without an uncountable independent set. There are S and \mathcal{B} which satisfy the conditions in 2.6, if and only if \mathbf{G} is not countably sparse.*

PROOF OF LEMMA 3.5. Assume first that S and \mathcal{B} satisfy the conditions in 2.6 and let $Q \subseteq S$, $H \subseteq \omega_1$ be countable. Put $\delta = \sup(Q \cup H)$ and choose $b \in \mathcal{B}$ so that $\delta < \min b$. By our assumption, for each $s \in S \cap \delta$, we have $(\{s\} \times b) \cap \mathbf{G} \neq \emptyset$, and since $Q \subseteq S \cap \delta$, S witnesses that \mathbf{G} is not countably sparse.

Now assume that \mathbf{G} is not countably sparse. Let X be an uncountable subset of ω_1 which witnesses it. In particular we have:

$$(1) \quad \forall \delta \in \omega_1, \exists b \in [\omega_1]^{<\omega} \text{ such that } \min b > \delta \\ \text{and } \forall \gamma \in X \cap \delta, (\{\gamma\} \times b) \cap \mathbf{G} \neq \emptyset.$$

Define by transfinite induction sequences $\langle \delta_\xi \rangle_{\xi \in \omega_1}$ of countable ordinals and $\langle b_\xi \rangle_{\xi \in \omega_1}$ of finite subsets of ω_1 , subject to the following conditions for all $\xi \in \omega_1$:

- (a) $\min b_\xi > \delta_\xi$ and $\forall \gamma \in X \cap \delta_\xi, (\{\gamma\} \times b_\xi) \cap \mathbf{G} \neq \emptyset$,
- (b) $\delta_{\xi+1} = \max b_\xi$, and $\delta_\xi = \sup \{\delta_\tau : \tau < \xi\}$ if ξ is a limit ordinal.

(1) above implies that our construction can be performed. Clearly $S = \{\delta_\xi : \xi \in \omega_1\}$ and $\mathcal{B} = \{b_\xi : \xi \in \omega_1\}$ satisfy the conditions in 2.6. \square

COROLLARY 3.6. *Axiom S is equivalent to the Countably Sparse Graph Axiom.* \square

THEOREM 3.7. *Axiom S is equivalent to Axiom 1.*

PROOF OF THEOREM 3.7.

LEMMA 3.7.1. *The Countably Sparse Graph Axiom implies Axiom 1.*

PROOF OF LEMMA 3.7.1. Assume the Countably Sparse Graph Axiom, and let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ be a collection of countable subsets of ω_1 , such that for every

uncountable $X \subseteq \omega_1$, there exists a countable set $Q \subseteq X$ which cannot be covered by a finite subfamily of \mathcal{S} . Define a graph \mathbf{G} on ω_1 by

$$(1) \quad \langle \alpha, \beta \rangle \in \mathbf{G} \quad \text{iff} \quad \alpha \neq \beta \quad \text{and} \quad (\alpha \in S_\beta \quad \text{or} \quad \beta \in S_\alpha).$$

Claim 3.7.2. \mathbf{G} is countably sparse.

Proof of Claim 3.7.2. Let $X \subseteq \omega_1$ be uncountable. Let $Q \subseteq X$ be a countable subset that cannot be covered by a finite subfamily of \mathcal{S} . Put $H = \bigcup \{S_\alpha : \alpha \in Q\}$. We will show that Q and H “work” for X (see 2.7). Clearly both Q and H are countable. Let b be a finite subset of $\omega_1 \setminus H$. By our assumption Q is not covered by $\bigcup \{S_\alpha : \alpha \in b\}$, so we can choose $\gamma \in \omega_1$ so that $\gamma \in Q \setminus \bigcup \{S_\alpha : \alpha \in b\}$. In particular

$$(2) \quad \forall \alpha \in b, \quad \gamma \notin S_\alpha.$$

Also, since b is disjoint from $H = \bigcup \{S_\alpha : \alpha \in Q\}$ and $\gamma \in Q$ we have

$$(3) \quad \forall \alpha \in b, \quad \alpha \notin S_\gamma.$$

Now (2) and (3) imply that $[\{\gamma\} \times b] \cap \mathbf{G} = \emptyset$. □

By the claim (see 2.8) there exists an uncountable \mathbf{G} -independent set $X \subseteq \omega_1$. We have $\forall \alpha, \beta \in X, \quad \alpha \neq \beta \implies \langle \alpha, \beta \rangle \notin \mathbf{G}$, thus $\forall \alpha \in X, \quad S_\alpha \cap X \subseteq \{\alpha\}$. □

LEMMA 3.7.3. *Axiom 1 implies the Countably Sparse Graph Axiom.*

PROOF OF LEMMA 3.7.3. Assume Axiom 1, and let \mathbf{G} be a countably sparse graph on ω_1 . For each $\alpha \in \omega_1$, let $S_\alpha = \{\beta < \alpha : \langle \alpha, \beta \rangle \in \mathbf{G}\}$.

Claim 3.7.4. $\mathcal{S} = \{S_\alpha : \alpha \in \omega_1\}$ satisfies the condition (b) in Lemma 3.1.

Proof of Claim 3.7.4. Let $X \subseteq \omega_1$ be uncountable. Choose countable sets $Q \subseteq X$ and $H \subseteq \omega_1$ so that for all finite sets $b \subseteq (\omega_1 \setminus H)$ there exists $\gamma \in Q$ such that $(\{\gamma\} \times b) \cap \mathbf{G} = \emptyset$ (see (2.7)). It is easy to see that Q cannot be covered by a finite subfamily of $\{S_\alpha : \alpha \notin H\}$. □

By Axiom 1, the claim and Lemma 3.1 there is an uncountable set $X \subseteq \omega_1$, such that for each $\alpha \in \omega_1$, $S_\alpha \cap X$ is a finite set. In particular

$$(4) \quad \forall \alpha \in X, \quad S_\alpha \cap X \quad \text{is a finite set.}$$

Let $X = \{\xi_\alpha : \alpha < \omega_1\}$, where $\alpha < \beta \implies \xi_\alpha < \xi_\beta$. Define a function $r : \text{Lim}(\omega_1) \rightarrow \omega_1$ by :

$$(5) \quad r(\alpha) = \min \{\gamma : X \cap S_{\xi_\alpha} \subseteq \xi_\gamma\} = \min \{\gamma : \forall \beta \geq \gamma, \quad \xi_\beta \notin S_{\xi_\alpha}\}.$$

By (4) and the fact that each S_α is a subset of α , $r : \text{Lim}(\omega_1) \rightarrow \omega_1$ is a well-defined regressive function. That is,

$$(6) \quad \forall \alpha \in \text{Lim}(\omega_1), \quad r(\alpha) < \alpha.$$

By the Pressing-down Lemma we can choose $\Lambda \in \omega_1$ with $r^{-1}(\Lambda)$ uncountable. We will show that $r^{-1}(\Lambda)$ is a \mathbf{G} -independent set. Pick $\xi_\alpha, \xi_\beta \in r^{-1}(\Lambda)$ with

$\xi_\beta < \xi_\alpha$. Now $r(\xi_\alpha) = r(\xi_\beta)$, hence (see (6)) $\xi_\beta > r(\xi_\alpha)$, and therefore (see(5)) $\xi_\beta \notin S_{\xi_\alpha}$, which since $\xi_\beta < \xi_\alpha$ implies $\{\xi_\alpha, \xi_\beta\} \notin \mathbf{G}$. □

This concludes the proof of the theorem. □

COROLLARY 3.8. *Axiom 1 is equivalent to the following axiom:*

Let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ be a collection of (countable) subsets of ω_1 , such that for every uncountable $X \subseteq \omega_1$, there exists a countable set $Q \subseteq X$ which cannot be covered by a finite subfamily of \mathcal{S} . Then there exists an uncountable set $X \subseteq \omega_1$, such that for each $\alpha \in X$, $S_\alpha \cap X \subseteq \{\alpha\}$.

PROOF OF COROLLARY 3.8. Clearly the above axiom implies Axiom 1. On the other hand our proof of Lemma 3.7.1 shows that the Countably Sparse Graph Axiom implies the above axiom. Hence by Corollary 3.6 and Theorem 3.7 the two axioms are equivalent. □

Note that an analogous modification of Axiom 0 leads to a statement which is clearly false in ZFC (to see it take \mathcal{S} to be the family of all finite subsets of ω_1).

Since Axiom 0 is clearly stronger than Axiom 1, the following observation is an easy consequence of Theorems 3.3 and 3.7.

COROLLARY 3.9. *TOP implies $\omega_1 \longrightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$.* □

Apparently it is not known whether the converse of 3.9 is true. In other words, *Question 3.10.* Does Axiom 1 imply Axiom 0?

A well-known piece of folklore is that the two axioms are equivalent under $\text{MA}(\omega_1)$. We will now show that a weakened version of $\text{MA}(\omega_1)$, together with Axiom 1, is enough to imply a strengthening of Axiom 0:

AXIOM 0*. Let A be a set of cardinality ω_1 and let $\{S_\alpha : \alpha \in A\}$ be a collection of (countable) sets. Then either

- (a) there is an uncountable subset Y of A , such that every countable $Q \subseteq Y$ is covered by finitely many S_α 's, or
- (b) $A = \bigcup \{X_n : n \in \omega\}$, where each $X_n \cap S_\alpha$ ($n \in \omega, \alpha \in A$) is finite.

First, let us observe that Axiom 1 is equivalent to the above axiom with (b) replaced by:

- (b') There is an uncountable subset X of A , such that $X \cap S_\alpha$ is finite for all $\alpha \in X$.

Indeed, Axiom 1 is clearly equivalent to this modification with $A = \omega_1$ and $S_\alpha \subseteq \omega_1$ for all α . Conversely, assuming Axiom 1 we can easily replace ω_1 everywhere in Axiom 1 by any set of cardinality ω_1 . Also, S_α need not be a subset of A nor ω_1 : conclusions (a), (b) and (b') hold iff they hold for all $S_\alpha \cap A$ in place of S_α .

We will use the following weakening of $\text{MA}(\omega_1)$.

3.11. $\text{MA}(\omega_1; \text{precaliber } \aleph_1)$. If \mathbf{P} is a p.o. set with precaliber \aleph_1 downwards, then for every family \mathcal{D} of ω_1 \downarrow -dense subsets of \mathbf{P} , there is a set that is \mathbf{P} -generic for \mathcal{D} .

It is known that $\text{MA}(\omega_1; \text{precaliber } \aleph_1)$ is strictly weaker than $\text{MA}(\omega_1)$; for details see [W] or [F].

THEOREM 3.12. *$\text{MA}(\omega_1; \text{precaliber } \aleph_1)$ and Axiom 1 together imply Axiom 0*.*

PROOF OF THEOREM 3.12. Assume $\text{MA}(\omega_1; \text{precaliber } \aleph_1) + \text{Axiom 1}$. Let A be an uncountable set, and let $\mathcal{S} = \{S_\alpha : \alpha \in A\}$ be a collection of sets such that (a) in Axiom 0* fails to hold. We may assume that $A \subseteq \omega_1$ and that $S_\alpha \subseteq A$ for each α . Let \mathbf{P} be a set of all ordered pairs $\langle p, a \rangle$, where $p : F \rightarrow \omega$ is a function

with $F = \text{dom}(p) \in [A]^{<\omega}$, and $a \in [A]^{<\omega}$. Let $\leq_{\mathbf{P}}$ be a relation on \mathbf{P} given by:

(1)

$\langle q, b \rangle \leq_{\mathbf{P}} \langle p, a \rangle$ iff

$p \subseteq q$, $a \subseteq b$ and $\forall \alpha \in \bigcup \{S_\xi : \xi \in a\} \cap (\text{dom}(q) \setminus \text{dom}(p))$, $q(\alpha) \notin \text{ran}(p)$.

Clearly $\langle \mathbf{P}, \leq_{\mathbf{P}} \rangle$ is a p.o. set.

LEMMA 3.12.1. $\langle \mathbf{P}, \leq_{\mathbf{P}} \rangle$ has precaliber \aleph_1 .

PROOF OF LEMMA 3.12.1. Let $B = \langle p_\alpha, a_\alpha \rangle_{\alpha < \omega_1}$ be an uncountable subset of \mathbf{P} . If for some p the set $E_p = \{\alpha < \omega_1 : p_\alpha = p\}$ is uncountable, then $\langle p_\alpha, a_\alpha \rangle_{\alpha \in E_p}$ is an uncountable \downarrow -centered subset of B . Otherwise, the Δ -system Lemma implies that there is an uncountable $X \subseteq \omega_1$, an $n \in \omega$ and $r \in [A]^{<\omega}$ such that:

(2)

$|\text{dom}(p_\alpha)| = n$ for all $\alpha \in X$,

(3)

$\langle \text{dom}(p_\alpha) \rangle_{\alpha \in X}$ are all distinct, and they form a Δ -system with root r ,

(4)

$\langle p_\alpha \rangle_{\alpha \in X}$ agree on r .

Let $m = n - |r|$. Clearly $m > 0$. For $\xi \in X$ and $i = 1, \dots, m$ let $b_i(\xi)$ be the i th element of $\text{dom}(p_\xi) \setminus r$, and let $T_\xi = \bigcup \{S_\gamma : \gamma \in a_\xi\}$. For each $\alpha = b_i(\xi)$ let $V_\alpha = T_\xi$. Note that we never have $b_i(\xi) = b_j(\eta)$, if $i \neq j$ or $\xi \neq \eta$, so V_α is well defined for all α .

Define by induction uncountable subsets X_0, \dots, X_m of X as follows: let $X_0 = X$, and if X_i has been defined for some $i < m$, then since it is uncountable, (a) of Axiom 0^* continues to fail with S_α replaced by V_α and A replaced by $\{b_{i+1}(\xi) : \xi \in X_i\}$, hence by (b') we can choose an uncountable $X_{i+1} \subseteq X_i$ such that $V_\alpha \cap \{b_{i+1}(\xi) : \xi \in X_{i+1}\}$ is finite for all $\alpha \in X_{i+1}$.

Each T_ξ , $\xi \in X_m$, meets $\{b_i(\eta) : 1 \leq i \leq m, \eta \in X_m\}$ in a finite set, and hence meets only finitely many $\text{dom}(p_\eta) \setminus r$, $\eta \in X_m$. By the Δ -system lemma there is an uncountable $Y \subseteq X_m$, such that if $C = \bigcup \{\text{dom}(p_\eta) : \eta \in Y\} \setminus r$, then $\langle C \cap T_\xi \rangle_{\xi \in Y}$ forms a Δ -system with root R . Let $Z = \{\eta \in Y : \text{dom}(p_\eta) \cap R = \emptyset\}$. Since R is disjoint from r , Z is uncountable. Let $D = \bigcup \{\text{dom}(p_\eta) : \eta \in Z\} \setminus r$. Clearly

(5)

$\langle D \cap T_\xi \rangle_{\xi \in Z}$ is a pairwise disjoint family.

Define by transfinite induction a sequence $\langle x_\alpha \rangle_{\alpha < \omega_1}$ of distinct elements of Z as follows: let x_0 be arbitrary; if $\langle x_\alpha \rangle_{\alpha < \beta}$ have been defined, let

(6)

$x_\beta = \min \left\{ \gamma \in Z : (\text{dom}(p_\gamma) \setminus r) \cap \bigcup_{\alpha < \beta} T_{x_\alpha} = T_\gamma \cap \bigcup_{\alpha < \beta} (\text{dom}(p_{x_\alpha}) \setminus r) = \emptyset \right\}$.

Since $\bigcup \{T_{x_\alpha} : \alpha < \beta\} \cap D$ and $\bigcup \{\text{dom}(p_{x_\alpha}) : \alpha < \beta\} \setminus r$ are both countable, (3) and (5) imply that x_β is well defined. Now, (6) implies that $(\text{dom}(p_{x_\alpha}) \setminus r) \cap T_{x_\beta} = \emptyset$ whenever $\alpha \neq \beta$, hence $\langle \langle p_{x_\alpha}, a_{x_\alpha} \rangle : \alpha < \omega_1 \rangle$ is an uncountable \downarrow -centered subset of B (see (1)). \square

For each $\alpha \in A$, let $D_\alpha = \{\langle p, a \rangle : \alpha \in \text{dom}(p) \text{ and } \alpha \in a\}$. Then D_α is \downarrow -dense for each α . Let G be \mathbf{P} -generic for $\mathcal{D} = \{D_\alpha : \alpha \in A\}$, and let $f = \bigcup \{p : \langle p, a \rangle \in G \text{ for some } a\}$. Then f is a function from A to ω , such that $f^{-1}\{n\} \cap S_\alpha$ is finite for all n and all α . Indeed if $\langle p, a \rangle \in G \cap D_\alpha$, and $n \in \text{ran}(p)$, then $f^{-1}\{n\} \cap S_\alpha = p^{-1}\{n\} \cap S_\alpha$. Of course $A = \bigcup \{f^{-1}\{n\} : n \in \omega\}$. \square

COROLLARY 3.13. *If $\text{MA}(\omega_1; \text{precaliber } \aleph_1)$, then Axioms 0, 1 and 0^* are equivalent to each other and to the following axiom:*

Let $\{S_\alpha : \alpha \in \omega_1\}$ be a collection of (countable) sets, and let A be any set. Then either

- (a) *there is an uncountable subset Y of A , such that every countable $Q \subseteq Y$ is covered by a finitely many S_α 's, or*
- (b) *$A = \bigcup \{X_n : n \in \omega\}$, where each $X_n \cap S_\alpha$ ($n \in \omega, \alpha \in \omega_1$) is finite.*

PROOF OF COROLLARY 3.13. By 3.9 and 3.12, Axioms 0, 1 and 0^* are equivalent under $\text{MA}(\omega_1; \text{precaliber } \aleph_1)$. Of course, Axiom 0^* is equivalent to the case “ A is uncountable” of the above axiom. But if A is countable then the above axiom holds in ZFC because of (b). Thus the above axiom is equivalent to Axiom 0^* in ZFC. \square

Question 3.14. Do any of Axioms 0, 1 or 0^* imply $\text{MA}(\omega_1; \text{precaliber } \aleph_1)$?

An affirmative answer for Axiom 1 would imply it for the other two axioms, and also an affirmative answer to Question 3.10. We do not even know whether Axiom 0^* implies $\text{MA}(\omega_1; \sigma\text{-centered})$, which is equivalent to the axiom $\mathfrak{p} > \omega_1$, as shown by M. Bell in [W, 5.16]. We do however know that Axiom 1 implies $\mathfrak{b} > \omega_1$, since Todorćević has shown that $\mathfrak{b} = \omega_1$ implies the existence of S -spaces. For more on \mathfrak{p} , \mathfrak{b} , and other “small” uncountable cardinals, see [vD] or [V].

§4. Applications of Axioms 0 and 1. The first application of TOP (Axiom 0) will be to the theory of directed sets. This application was given by Baumgartner in [B, Theorem 5.10]. The proof given by Baumgartner uses $\text{MA}(\omega_1)$ as an additional hypothesis and is much longer than the one below. Recall that a p.o. set \mathbf{D} is \downarrow -directed iff \mathbf{D} is an \downarrow -directed set in itself (see 2.1).

THEOREM 4.1 ([Devlin and Steprans, Baumgartner]). *Assume Axiom 0, and let $\langle \mathbf{D}, \leq_{\mathbf{D}} \rangle$ be a \downarrow -directed p.o. set of cardinality ω_1 . If every uncountable subset of \mathbf{D} contains a countable unbounded from below set, then there exists an uncountable subset of \mathbf{D} every infinite subset of which is unbounded from below.*

PROOF OF THEOREM 4.1. Let $\mathbf{D} = \{d_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$ put $S_\alpha = \{\beta < \omega_1 : d_\beta \geq_{\mathbf{D}} d_\alpha\}$, and let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$. We show that \mathcal{S} satisfies the hypothesis of Axiom 0. Let $X \subseteq \omega_1$ be uncountable. Choose a countable unbounded from below set $Q \subseteq \{d_\alpha : \alpha \in X\}$, and let $Q' = \{\alpha \in \omega_1 : d_\alpha \in Q\}$. If for some finite $\{\alpha_1, \dots, \alpha_n\} \subseteq \omega_1$, we have $Q' \subseteq \bigcup \{S_{\alpha_i} : i \leq n\}$, then since $\langle \mathbf{D}, \leq_{\mathbf{D}} \rangle$ is \downarrow -directed, we can choose $\beta \in \omega_1$, so that $\forall i \leq n, d_{\alpha_i} \geq_{\mathbf{D}} d_\beta$, hence $Q' \subseteq \bigcup \{S_{\alpha_i} : i \leq n\} \subseteq S_\beta$, and Q is bounded from below (by d_β), which

contradicts our assumption. By Axiom 0 there exists an uncountable set $X \subseteq \omega_1$ such that

$$(1) \quad \forall \alpha \in \omega_1, S_\alpha \cap X \text{ is finite.}$$

Let $E = \{d_\alpha : \alpha \in X\}$. If $F \subseteq E$ is infinite, then so is $Y = \{\alpha : d_\alpha \in F\} \subseteq X$. By (1) we have $\forall \alpha \in \omega_1, Y \not\subseteq S_\alpha$. Hence $\forall \alpha \in \omega_1, F \not\subseteq \{d_\beta : d_\beta \geq_D d_\alpha\}$, and F is unbounded from below. \square

The other applications of Axioms 0 and 1 are to the theory of S -spaces and locally countable regular spaces. Recall that an S -space is a regular, hereditarily separable space which is not hereditarily Lindelöf. An elementary fact about S -spaces [R, Corollary 3.2] is that every one contains a locally countable subspace of cardinality ω_1 . [A space is said to be *locally countable* if every point has a countable neighborhood.] Of course such a subspace cannot be Lindelöf, and so it is also an S -space.

It was a major unsolved problem for many years whether S -spaces can be constructed from the usual (ZFC) axioms of set theory. Baumgartner and Todorčević independently and almost simultaneously showed that they can not. Here we give new proofs using Axioms 0 and 1.

Call a space ω -fair if every countable subset has countable closure. Obviously, an uncountable ω -fair space is not separable, and so the following theorem implies that there are no S -spaces under Axiom 0.

THEOREM 4.2. *Axiom 0 implies that every locally countable regular space of cardinality ω_1 has an uncountable closed ω -fair subspace.*

PROOF OF THEOREM 4.2. Let the space have ω_1 as an underlying set. For each $\alpha \in \omega_1$, let S_α be an open neighborhood of α with countable closure. If there is an uncountable subset Z of ω_1 which meets every S_α in a finite set, then any such Z is clearly a closed discrete subspace, hence ω -fair. If there is no such Z , then by Axiom 0 there is an uncountable $X \subseteq \omega_1$ such that every countable subset of X is contained in a finite union of S_α 's, hence has countable closure in the whole space. The closure of X is the desired subspace, since every countable subspace of \bar{X} is in the closure of a countable subspace of X , hence has countable closure in \bar{X} . \square

If we use Axiom 1 instead, the above argument gives a conclusion easily seen equivalent to the nonexistence of S -spaces. Recall that a *free sequence* in a space Y is a transfinite sequence $\langle y_\alpha : \alpha < \tau \rangle$ in Y with the property that, for each $\gamma < \tau$, the closure of $\{y_\alpha : \alpha < \gamma\}$ in Y does not meet that of $\{y_\delta : \delta \geq \gamma\}$. Clearly, every closed discrete subspace is a free sequence in any one-to-one well-ordering, and every free sequence is a discrete subspace.

THEOREM 4.3. *Axiom 1 implies that every locally countable regular space of cardinality ω_1 has an uncountable discrete subspace.*

PROOF OF THEOREM 4.3. We follow the preceding proof, except that now Z is an uncountable subset of ω_1 such that for each $\alpha \in Z$, $S_\alpha \cap Z$ is a finite set. This makes Z into a discrete subspace. If no such Z exists, we get the closed ω -fair $\bar{X} = Y$ as before, and use:

LEMMA. *Every locally countable, uncountable, ω -fair space has an uncountable free sequence.*

PROOF OF THE LEMMA. Let Y be locally countable, uncountable and ω -fair. Define by transfinite induction a sequence $\langle y_\alpha \rangle_{\alpha \in \omega_1}$ of distinct elements of Y as follows: let y_0 be arbitrary; if $\langle y_\alpha \rangle_{\alpha < \gamma}$ have been defined, let Y_γ be a countable open subset of Y containing the (countable) closure of $\{y_\alpha : \alpha < \gamma\}$, and let y_γ be any point of $Y \setminus \bigcup \{Y_\alpha : \alpha \leq \gamma\}$. Then the closure of $\{y_\alpha : \alpha < \gamma\}$ is a subset of Y_γ which in turn misses the closure of $\{y_\delta : \delta \geq \gamma\}$. \square

COROLLARY 4.4. *Axioms 0 and 1 each imply there are no S -spaces.* \square

Applying the lemma used in Theorem 4.3 to Theorem 4.2, we get:

COROLLARY 4.5. *Axiom 0 implies that every locally countable regular space of cardinality ω_1 has an uncountable free sequence.* \square

Theorem 4.2 does not make full use of the machinery in the proof. It is easy to see, in fact, that the proof establishes the Axiom 0 part of the following theorem:

THEOREM 4.6. *Assume Axiom 0 and let Y be a locally countable regular space of cardinality ω_1 . Then either*

- (a) *Y has an uncountable closed discrete subspace, or*
- (b) *Y has an uncountable closed subspace K , such that each countable subset of K is contained in an open subspace of Y with countable closure in Y .*

Moreover, if we assume Axiom 0, then (a) can be strengthened to:*

- (a*) *Y is a countable union of closed discrete subspaces.* \square

QUESTION 4.7. Are any of the topological consequences of Axiom 0 in 4.2, 4.5, or 4.6 implied by (hence equivalent to) the nonexistence of S -spaces?

If even one of them is not implied by the nonexistence of S -spaces, this would have the consequence that Axiom 1 is not equivalent to Axiom 0, answering Question 3.10 negatively.

REFERENCES

- [B] JAMES E. BAUMGARTNER, *Applications of the Proper Forcing Axiom*, **Handbook of set-theoretic topology** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 913-959.
- [VD] ERIC K. VAN DOUWEN, *The integers and topology*, **Handbook of set-theoretic topology** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 111-167.
- [F] D. H. FREMLIN, *Consequences of Martin's axiom*, Cambridge University Press, Cambridge, 1984.
- [N] PETER NYIKOS, *Applications of the Proper Forcing Axiom*, preprint.
- [R] JUDY ROITMAN, *Basic S and L* , **Handbook of set-theoretic topology** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 295-326.
- [T] STEVO TODORČEVIĆ, *Forcing positive partition relations*, **Transactions of the American Mathematical Society**, vol. 280 (1983), pp. 703-720.
- [V] JERRY E. VAUGHAN, *Small uncountable cardinals and topology. With an appendix by S. Shelah*, **Open problems in topology** (J. van Mill and G. M. Reed, editors), North-Holland, Amsterdam, 1990, pp. 195-218.
- [W] WILLIAM WEISS, *Versions of Martin's axiom*, **Handbook of set-theoretic topology** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 827-886.

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