12-1986

Alfred Tarski and Undecidable Theories

George F. McNulty

University of South Carolina - Columbia, mcnulty@math.sc.edu

Publication Info

Published in *Journal of Symbolic Logic*, Volume 51, Issue 4, 1986, pages 890-898.

THE JOURNAL OF SYMBOLIC LOGIC
Volume 51, Number 4, Dec. 1986

ALFRED TARSKI AND UNDECIDABLE THEORIES

GEORGE F. MCNULTY

Alfred Tarski identified decidability within various logical formalisms as one of the principal themes for investigation in mathematical logic. This is evident already in the focus of the seminar he organized in Warsaw in 1926. Over the ensuing fifty-five years, Tarski put forth a steady stream of theorems concerning decidability, many with far-reaching consequences. Just as the work of the 1926 seminar reflected Tarski's profound early interest in decidability, so does his last work, ***A formalization of set theory without variables***, a monograph written in collaboration with S. Givant [8–$^{\mathrm{m}}$]. An account of the Warsaw seminar can be found in Vaught [1986].

Tarski's work on decidability falls into four broad areas: elementary theories which are decidable, elementary theories which are undecidable, the undecidability of theories of various restricted kinds, and what might be called decision problems of the second degree. An account of Tarski's work with decidable elementary theories can be found in Doner and van den Dries [1987] and in Monk [1986] (for Boolean algebras). Vaught [1986] discusses Tarski's contributions to the method of quantifier elimination. Our principal concern here is Tarski's work in the remaining three areas.

We will say that a set of elementary sentences is a *theory* provided it is closed with respect to logical consequence and we will say that a theory is *decidable* or *undecidable* depending on whether it is a recursive or nonrecursive set. The notion of a theory may be restricted in a number of interesting ways. For example, an *equational theory* is just the set of all universal sentences, belonging to some elementary theory, whose quantifier-free parts are equations between terms. Another example is obtained by considering just the atomic sentences belonging to some elementary theory. These "atomic" theories play a central role with respect to presentations of algebras and their word problems. The notion of undecidability can be referred to equational or atomic theories as well as to elementary theories. An example of a decision problem of the second degree is the question of whether, for a fixed elementary language, the collection of finite consistent sets of sentences is recursive. Tarski's most significant and best-known work on undecidability centers on elementary theories. His work on the undecidability of certain equational theories has yet to appear in full; it is of considerable interest.

Received August 15, 1985; revised January 9, 1986.

0022-4812/86/5104-0004/$01.90

It is possible to discern an insight which underlies Tarski's approach to decidability problems for elementary (and restricted) theories. Roughly speaking, Tarski's results proceed from an analysis of definability within the various relevant theories. Thus, Tarski's approach is fundamentally semantical, taking advantage of the expressive power of the formalisms at hand. The notion of interpretability—of definability of the fundamental notions of one theory in another—is a key ingredient in Tarski's work. Indeed, results about the undecidability of various theories are only one kind of consequence of Tarski's work with the notion of interpretability; in certain cases undecidability was not the principal goal. Some results of this kind are discussed below, even though they were motivated by concerns other than decidability.

Perhaps Tarski's most far-reaching contribution to our understanding of undecidable theories is the paper *A general method in proofs of undecidability*, which is the first of the three papers that comprise the book ***Undecidable theories*** by Tarski, Mostowski, and Robinson [53m]. The material contained in this paper originated during 1938–1939, hard upon the first discovery of undecidable elementary theories by Church [1936] and [1936a] and by Rosser [1936]. Building on the celebrated results of Gödel [1931], Church had shown that elementary Peano arithmetic and certain of its subtheories, notably the logical validities for the language of arithmetic, are undecidable. Rosser advanced the work begun by Church and proved that elementary Peano arithmetic has the property that each of its consistent extensions is also undecidable. Theories, like elementary Peano arithmetic, with this property were called by Tarski *essentially undecidable theories.*

The method described by Tarski has become well known under the name of the "method of interpretation". The method used by Church and Rosser (and later by others, including Tarski) might be called the "direct method". In essence, the direct method proceeds by faithfully coding one of the many versions of the notion of recursive function into the target theory, the theory to be shown undecidable. Once this is accomplished, the algorithmic unsolvability of, let us say, the halting problem can be invoked. The direct method is difficult, even forbidding, in most cases. It tends to rely on a combinatorial or syntactical emulation which may carry no particular meaning within the theory at hand. In contrast, the method of interpretation makes effective use of the expressive power of elementary languages in order to reduce the undecidability of one theory to that of another. To illustrate the method, consider the theory $\mathrm{Th}\langle\omega, +, \cdot, \leq\rangle$ of the natural numbers and the theory $\mathrm{Th}\langle Z, +, \cdot\rangle$ of the ring of integers. The first is undecidable, according to the work of Church and Rosser. By a well-known theorem of Lagrange, an integer is nonnegative if and only if it can be written as a sum of four squares. This can be readily expressed by a formula $\psi(x)$ in the elementary language of rings. The same applies to the order relation: $\exists u[\psi(u) \wedge (x + u = y)]$ is an elementary formula in the language of rings which, in $\langle Z, +, \cdot\rangle$, defines the order relation. So it is possible to associate with each sentence ϕ, in the language of the natural numbers, a new sentence ϕ^*, in the language of rings, which is obtained from ϕ by using the formula written out above in place of $x \leq y$ and by restricting the variables to range over the set defined by $\psi(x)$. Clearly, ϕ is true of $\langle\omega, +, \cdot, \leq\rangle$ if and only if ϕ^* is true of $\langle Z, +, \cdot\rangle$. Hence the elementary theory of the ring of integers must be undecidable.

In order to establish the undecidability of a theory T it suffices to show that some essentially undecidable theory T_0 can be interpreted, not necessarily in T, but (what is much easier) in some consistent extension of T—provided only that T_0 is finitely axiomatizable. This implies that a consistent extension T'_0 of T_0 can be interpreted in a consistent extension T' of T that is finitely axiomatizable relative to T. T'_0 is undecidable since T_0 is essentially undecidable. Thus T' is undecidable, and the undecidability of T follows by the deduction theorem from the fact that T' is finitely axiomatizable relative to T.

This method opened the way for many applications, the earliest due to Tarski himself. In fact, using his method, Tarski established the undecidability of the elementary theories of groups [49ªh] (expounded fully in *The undecidability of the elementary theory of groups*, which is the third paper in Tarski, Mostowski, and Robinson [53ᵐ]), of lattices, of modular lattices, of complemented modular lattices, of abstract projective geometries, and of various theories of rings; see Tarski [49ªf], [49ªi] and Mostowski and Tarski [49ªg]. W. Szmielew and Tarski [52b] established the essential undecidability of various weak consistent fragments of set theory. Some years later, Tarski [59] and Tarski and Szczerba [79] offered undecidability results in various branches of elementary geometry.

In order to use the method of interpretation, as originally conceived by Tarski, it is necessary to have a finitely axiomatizable, essentially undecidable theory which is weak enough in its own means of expression to hold out the hope that it can be interpreted into theories quite distant from it. At least the first such theory must arise by some means other than the method of interpretation itself. In the late 1930's, only Peano arithmetic and its consistent extensions as well as various versions of set theory were known to be essentially undecidable. However, it was known that Peano arithmetic is not finitely axiomatizable and, even disregarding its possible inconsistency, set theory is so rich in its mathematical content that it seems difficult to interpret it into any wide variety of other theories. Mostowski and Tarski constructed, in 1939, a fairly weak, finitely axiomatizable fragment of arithmetic which is essentially undecidable. Around 1949/50, Raphael Robinson and Tarski simplified this fragment and the whole construction was put on the very general footing found in *Undecidability and essential undecidability in arithmetic*, which is the second paper in Tarski, Mostowski, and Robinson [53ᵐ].

They use the direct method to demonstrate that their small fragment Q of elementary arithmetic is essentially undecidable. Because Q is finitely axiomatizable, it follows by the deduction theorem that any theory compatible with Q must be undecidable. In this way, Tarski, Mostowski, and Robinson found a tremendous extension of the earlier results of Church and Rosser. The form of the direct method employed by Mostowski, Robinson, and Tarski is based on ideas found in Tarski [33ᵐ] and [35b] and in Gödel [1931]. In the general scheme which they set up and afterwards apply to their fragment of arithmetic, they presuppose an elementary language equipped with denumerably many terms $\Delta_0, \Delta_1, \ldots$ in which no variables occur and an effective scheme for numbering the formulas and other syntactical structures of the language. They say a set P of natural numbers is *definable* in the theory T provided there is a formula $\phi(x)$ such that $T \vdash \phi(\Delta_n)$ if $n \in P$ and $T \vdash \neg\phi(\Delta_n)$ otherwise. The notion of a definable function is similar. They then introduce the well-known diagonal function D. According to a celebrated result of Tarski [33$^{\mathrm{m}}$] and [35b] it cannot happen that both the diagonal function D and the set of numbers of the theorems of T are definable. (See Vaught [1986] for a discussion of this important result.) Since the diagonal function is recursive, the essential undecidability of T follows, provided every recursive function is definable in T. Mostowski, Robinson, and Tarski base their demonstration that every recursive function is definable in Q on the characterization of recursive functions found by Julia Robinson [1950]. Thus, as in the method of interpretation, Tarski's keen insight into the semantical notion of definability played a key role in the construction of an essentially undecidable finitely axiomatizable relatively weak fragment of arithmetic.

Soon after the publication of Tarski, Mostowski, and Robinson [53$^{\mathrm{m}}$] there was a sharp increase in research activity concerning undecidable theories. One center of activity had already emerged in Berkeley under the leadership of Tarski. The leitmotif of decidability can be discerned easily in the work of the Berkeley school during the 1950's. In Princeton, where Gödel was a member on the Institute and Church, Rosser, and Kleene had done their pioneering work in the theory of recursive functions, interest remained high in the undecidable, with a focus on, for example, word problems. W. W. Boone and M. Rabin were graduate students at Princeton during this time. Another great center emerged in Novosibirsk under the leadership of A. I. Mal′cev, a mathematician for whom Tarski had enormous respect. A number of Mal′cev's papers have been translated into English and collected in Mal′cev [1971]. The influence of Tarski's method of interpretation on Mal′cev's work is evident on even a superficial reading. At the hands of the Novosibirsk school and others in the Soviet Union, the method of interpretation was first refined and then recast in a more powerful form and many applications were obtained. These are gathered in the survey by Ershov, Lavrov, Taĭmanov, and Taĭtslin [1965], which also includes a number of sections taken from Tarski, Mostowski, and Robinson [53$^{\mathrm{m}}$] with little or no modification (thus making them widely available in Russian). A somewhat different and powerful variation on the method of interpretation, usually called the Rabin-Scott method, was described in Rabin [1965]. These techniques, as well as the direct method, are very much in use today. For example, using the Rabin-Scott variant, Burris and McKenzie [1981] have shown that the elementary theory of a congruence modular variety of algebras must be undecidable, unless the variety admits a very strong structure theorem, and McKenzie [1982], following the work of Zamjatin [1978], has shown that the elementary theory of any class V of groups is hereditarily undecidable, provided only that V contains all the subdirect powers of some nonabelian group.

The whole concept of interpretability has a long and intricate mathematical history. It is clearly present in Descartes' approach to geometry, and it can be seen in every coordinatization theorem since. It has been used to establish relative consistency results since the emergence of hyperbolic geometry. In Tarski's work, it is almost in the title of his monograph ***A decision method for elementary algebra and geometry*** [48$^{\mathrm{m}}$]. Givant and Tarski [77$^{\mathrm{a}}$a] point out the mutual interpretability of Peano arithmetic and the Zermelo-like theory of finite sets (which underlies, for example, our willingness to say that the Paris-Harrington [1977] discovery is a "natural" mathematical truth independent of Peano arithmetic). Szmielew and Tarski [49ªk] also concerns mutually interpretable theories. Recently, the notion of interpretability has been investigated in its own right with some interesting results. See, for example, Szczerba [1977], Mycielski [1977], and Garcia and Taylor [1984].

Let us now turn to Tarski's work concerning the undecidability of theories of a restricted kind. Using the method of interpretation, Tarski was able to produce a finitely axiomatizable equational theory which is essentially undecidable. He did this by interpreting set theory into an equational theory of relation algebras with additional distinguished individual constant symbols.

Tarski observed in *A general method in proofs of undecidability* that it is possible to interpret the quantifiers and connectives as well as the fundamental operation and relation symbols of a theory. When he made that observation, he had in hand an extensive manuscript (which was to grow into Tarski and Givant [8–ᵐ]) in which he had actually carried out such an interpretation. The underlying ideas are those of relation algebra and, more generally, algebraic logic, and can be traced back to Kuratowski and Tarski [31a] through Tarski [41]. (See Vaught [1986] and Jónsson [1986].) It should be stressed here that it is *interpretation* of the quantifiers and connectives, rather than a coding of them, which Tarski used.

Roughly speaking, an intended model of Tarski's axioms for relation algebra is a collection of binary relations on some set U which is closed with respect to the (Boolean) operations of intersection, union, and complementation and also with respect to the formation of converses and of the composition of two relations. The identity relation restricted to U is taken as a distinguished constant. In this way, the intention is that a relation algebra turn out to be an expansion of a Boolean set algebra. Tarski's plan for interpreting an elementary theory T_0 into an equational theory T of relation algebras, possibly with additional distinguish elements, was very natural: variables ranging over individuals and individual constant symbols of T, which are intended to represent binary relations, are used to interpret the predicates of T_0, the Boolean operation symbols of T interpret the connectives of T_0, the identity constant symbol of T interprets the equality symbol of T_0, and, finally, the operation symbol of T for composition of relations can be used to produce an interpretation of the existential quantifier. (Write out a definition of composition to see where the quantifier is!) There are two substantial barriers to the success of this plan. First, it has turned out that not all models of Tarski's axioms for relation algebras are representable as algebras of binary relations with fundamental operations as described above; see Lyndon [1950]. Second, the interpretation of the existential quantifier suggested above is not really powerful enough. It can only handle elementary sentences in which no more than three distinct variables occur. However, Tarski was able to show that if T_0 was powerful enough to provide the means to define a pairing function, then both of these difficulties could be overcome. For such an elementary theory T_0 Tarski could associate an equational theory T of relation algebras (with additional constants) and a map taking each elementary sentence ϕ of T_0 to an equation ϕ^* of T such that $T_0 \vdash \phi$ if and only if $T \vdash \phi^*$; moreover, T will be finitely axiomatizable if T_0 is. Tarski's line of reasoning here was carried out at the syntactical level, and it is of considerable complexity. Still, it uses several auxiliary theories and is very much along the lines of the simple procedure described in *A general method in proofs of undecidability*. R. Maddux [1978] and

[1978a], one of the last Ph.D. students of Tarski, subsequently discovered an elegant model-theoretic approach to these results. Tarski's method of interpreting elementary theories into equational theories applies, for example, to Peano arithmetic, to the system Q, and to virtually every version of set theory which has appeared in the literature. In this way, Tarski discovered what remains today as really the only example of an essentially undecidable equational theory. These results were announced in Tarski [53^ab], [53^ac], and [53^ad] and are fully expounded in the forthcoming monograph Tarski and Givant [8–^m].

Tarski's demonstration that certain equational theories of relation algebras are undecidable is an example of a restricted decision problem. Another example is the world problem for groups. It is possible to make a rough comparison of these two problems. Novikov [1955] and Boone [1954] discovered finitely presented groups which have recursively unsolvable word problems. This means that the language of group theory can be expanded by finitely many new constant symbols, and a finite set Δ of equations involving no variables can be constructed so that the set of all logical consequences of $\Gamma \cup \Delta$ which are equations without variables is not recursive, where Γ is the set of equations axiomatizing group theory. Tarski's result can be cast in the same form and one could say that he constructed a finitely presented relation algebra with an unsolvable word problem (but this would miss much of the point, since what Tarski did was offer an equational interpretation of set theory!). Now using the fact that Δ is a finite set of equations, the deduction theorem entails that the consequences of Γ, which have the form of quasi-identities (alias strict basic Horn sentences) without variables, also constitute a nonrecursive set. Finally, observing that the new constants do not occur in Γ, one can conclude that the universal Horn theory of groups is undecidable. In relation algebra, enough of the apparatus of the logical connectives remains, so that, loosely speaking, the argument just given can be conducted at the equational level, the result being a finitely axiomatizable undecidable equational theory of relation algebras. Notice that the essentially undecidable equational theory discussed above concerns relation algebras with additional individual constant symbols, and it is not an equational theory of relation algebras. Similar results have been obtained for cylindric algebras rather than relation algebras. See Henkin, Monk, and Tarski [71^m], [85^m], Henkin and Tarski [61a], and Maddux [1980]. Tarski was sharply aware that the meager means of expression available in such restricted settings makes results of this kind much more difficult, generally speaking, than the demonstration of the undecidability of full elementary theories. Tarski spoke highly of the results of Novikov and Boone, and later of the demonstration by R. Freese [1980] of the undecidability of the equational theory of modular lattices.

Here is a mathematical aside. It is easy to see that the equational theory axiomatized by $\Gamma \cup \Delta$ above is undecidable. It is a theory of groups with additional distinguished elements. Of course the same reasoning applies to the earlier results of Markov [1947] and Post [1947] concerning the word problem for semigroups. Tarski recalled being firmly convinced, when he announced his results in 1953, that no earlier published results concerned undecidable equational theories. A decade later A. I. Mal′cev [1966] appeared, in which it is again announced that for the first time an undecidable finitely axiomatizable equational theory had been discovered. Mal′cev even makes use of the Markov-Post result in his construction. Mal′cev was unaware of Tarski [53ᵃc]. It was not until the late 1960's that Tarski observed the connection between word problems and equational theories.

The fourth sort of contribution Tarski made in the area of undecidability concerns decision problems of the second degree. These problems center on the existence of algorithms for discerning of each theory in a given class whether or not it has a given property. For example, is there an algorithm for determining whether elementary theories are consistent? Now an elementary theory is an infinite object and is, therefore, unsuitable as an input of an algorithm. To avoid problems of this kind, Tarski imposed a restriction to finite sets. For example, given a fixed finite language, is the collection of all finite consistent sets of sentences recursive? What about the collection of all finite sets of sentences which axiomatize decidable elementary theories? In *A general method in proofs of undecidability*, Tarski observes that the answer to both of these questions is no, provided only that some finitely axiomatizable essentially undecidable theory can be formulated in the language. The simple proof relies heavily on the availability of negation and the use of the deduction theorem. Problems of this kind, when addressed to finite presentations of semigroups instead of finite sets of sentences, had been investigated by Markov [1951a,b]. In the setting of group presentations, they were investigated by Adjan [1958] and Rabin [1958]. The results obtained, both for semigroups and groups, are quite powerful. Tarski had raised similar problems earlier with regard to the propositional calculus. For example, at the Princeton Bicentennial in 1946 he asked whether the collection of finite sets of propositions which could axiomatize propositional logic (using modus ponens and substitution as the only rules of inference) was recursive. Linial and Post [1949] announced that the solution was negative. See Yntema [1964] and Singletary [1968] for later results in this direction. In Tarski [68], a number of problems of this kind were raised in connection with equational logic. Subsequently, these have been substantially resolved by Perkins [1966], Murskiĭ [1971], McNulty [1972], Pigozzi [1976], and McKenzie [1975]. However, one of the most outstanding open problems in equational logic, which was posed by Tarski in the mid-1960's, remains to be solved: Is there an algorithm which, upon input of a finite algebra, will determine whether its equational theory is finitely axiomatizable?

Mathematics, in the end, is a work of the human spirit as much as it is a work of the intellect. For me, Alfred Tarski's greatest contributions to the advancement of the mathematical enterprise arose from the vitality of his engagement with the full range of human experience, from his generosity of mind and spirit, from his genuine interest in the lives of other people, and from his warm hospitality and great personal charm. Because of attributes like these, Alfred Tarski attracted, supported, and encouraged what seems like whole generations of talented people throughout the world, to the benefit of mathematical logic.

REFERENCES
(OTHER THAN WORKS OF TARSKI)

S. I. ADJAN

[1958] *On algorithmic problems in effectively complete classes of groups*, ***Doklady Akademii Nauk SSSR***, vol. 123, pp. 13–16. (Russian)

W. W. BOONE

[1954] *Certain simple unsolvable problems of group theory*, ***Indagationes Mathematicae***, vol. 16, pp. 231–237.

S. BURRIS and R. MCKENZIE

[1981] ***Decidability and Boolean representations***, *Memoirs of the American Mathematical Society*, no. 246, American Mathematical Society, Providence, Rhode Island.

A. CHURCH

[1936] *A note on the Entscheidungsproblem*, this JOURNAL, vol. 1, pp. 40–41, 101–102.

[1936a] *An unsolvable problem of elementary number theory*, ***American Journal of Mathematics***, vol. 58, pp. 345–363.

J. DONER and L. VAN DEN DRIES

[1987] This JOURNAL, vol. 52 (to appear).

YU. L. ERSHOV, I. A. LAVROV, A. D. TAĬMANOV and M. A. TAĬTSLIN

[1965] *Elementary theories*, ***Uspehi Matematičeskih Nauk***, vol. 20, no. 4 (124), pp. 37–108; English translation, ***Russian Mathematical Surveys***, vol. 20, no. 4, pp. 35–105.

R. FREESE

[1980] *Free modular lattices*, ***Transactions of the American Mathematical Society***, vol. 261, pp. 81–92.

O. GARCIA and W. TAYLOR

[1984] ***The lattice of interpretability types of varieties***, *Memoirs of the American Mathematical Society*, no. 305, American Mathematical Society, Providence, Rhode Island.

K. GÖDEL

[1931] *Über formal unentscheidbare Sätze der* ***Principia Mathematica*** *und verwandter Systeme*. I, ***Monatshefte für Mathematik und Physik***, vol. 38, pp. 173–198.

B. JÓNSSON

[1986] *The contributions of Alfred Tarski to general algebra*, this JOURNAL, vol. 51, pp. 883–889.

S. LINIAL and E. POST

[1949] *Recursive unsolvability of the deducibility, Tarski's completeness and independence of axioms problems of propositional calculus*, ***Bulletin of the American Mathematical Society***, vol. 55, p. 50.

R. C. LYNDON

[1950] *The representation of relational algebras*, ***Annals of Mathematics***, ser. 2, vol. 51, pp. 707–729.

R. MADDUX

[1978] ***Topics in relation algebras***, Ph.D. Thesis, University of California, Berkeley, California.

[1978a] *Some sufficient conditions for the representability of relation algebras*, ***Algebra Universalis***, vol. 8, pp. 162–172.

[1980] *The equational theory of CA_3 is undecidable*, this JOURNAL, vol. 45, pp. 311–316.

A. I. MAL'CEV

[1966] *Identical relations on varieties of quasigroups*, ***Matematičeskiĭ Sbornik***, vol. 69 (111), pp. 3–12; English translation, [1971], Chapter XXIX, pp. 384–395.

[1971] ***The metamathematics of algebraic systems: collected papers 1936–1967*** (B. F. Wells, III, editor), North-Holland, Amsterdam.

A. A. MARKOV

[1947] *On the impossibility of certain algorithms in the theory of associative systems*. I, II, ***Doklady Akademii Nauk SSSR***, vol. 55, pp. 587–590; vol. 58, pp. 353–356; English translation of I, ***Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS***, vol. 55, pp. 583–586.

[1951a] *Impossibility of certain algorithms in the theory of associative systems*, ***Doklady Akademii Nauk SSSR***, vol. 77, pp. 19–20. (Russian)

[1951b] *The impossibility of algorithms for the recognition of certain properties of associative systems*, ***Doklady Akademii Nauk SSSR***, vol. 77, pp. 953–956. (Russian)

R. MCKENZIE

[1975] *On spectra, and the negative solution of the decision problem for identities having a finite nontrivial model*, this JOURNAL, vol. 40, pp. 186–196.

[1982] *Subdirect powers of nonabelian groups*, ***Houston Journal of Mathematics***, vol. 8, pp. 389–399.

G. MCNULTY

[1972] ***The decision problem for equational bases of algebras***, Ph.D. Thesis, University of California, Berkeley, California.

J. D. MONK

[1986] *The contributions of Alfred Tarski to algebraic logic*, this JOURNAL, vol. 51, pp. 899–906.

V. L. MURSKIĬ

[1971] *Nondiscernible properties of finite systems of identity relations*, ***Doklady Akademii Nauk SSSR***, vol. 196, pp. 520–522; English translation, ***Soviet Mathematics Doklady***, vol. 12, pp. 183–186.

J. MYCIELSKI

[1977] *A lattice of interpretability types of theories*, this JOURNAL, vol. 42, pp. 297–305.

P. S. NOVIKOV

[1955] *On the algorithmic undecidability of the word problem in groups*, ***Trudy Matematičeskogo Instituta imeni V. A. Steklova***, vol. 44, Izdatel'stvo Akademii Nauk SSSR, Moscow, 1955; English translation, ***American Mathematical Society Translations***, ser. 2, vol. 9, American Mathematical Society, Providence, Rhode Island, 1958, pp. 1–122.

J. B. PARIS and L. HARRINGTON

[1977] *A mathematical incompleteness in Peano's arithmetic*, ***Handbook of mathematical logic*** (J. Barwise, editor), North-Holland, Amsterdam, pp. 1113–1142.

P. PERKINS

[1966] ***Decision problems for equational theories of semigroups and general algebras***, Ph.D. Thesis, University of California, Berkeley, California.

D. PIGOZZI

[1976] *Base-undecidable properties of universal varieties*, ***Algebra Universalis***, vol. 6, pp. 193–223.

E. L. POST

[1947] *Recursive unsolvability of a problem of Thue*, this JOURNAL, vol. 12, pp. 1–11.

M. RABIN

[1958] *Recursive unsolvability of group theoretic problems*, ***Annals of Mathematics***. ser. 2, vol. 67, pp. 172–194.

[1965] *A simple method for undecidability proofs and some applications*, ***Logic, methodology and philosophy of science (proceedings, Jerusalem, 1964***; Y. Bar-Hillel, editor), North-Holland, Amsterdam, pp. 58–68.

J. ROBINSON

[1950] *General recursive functions*, ***Proceedings of the American Mathematical Society***, vol. 1, pp. 703–718.

J. B. ROSSER

[1936] *Extensions of some theorems of Gödel and Church*, this JOURNAL, vol. 1, pp. 87–91.

W. SINGLETARY

[1968] *Results regarding the axiomatization of partial propositional calculi*, ***Notre Dame Journal of Formal Logic***, vol. 9, pp. 193–221.

L. SZCZERBA

[1977] *Interpretability of elementary theories*, ***Logic, foundations of mathematics, and computability theory (proceedings, London, Ontario, 1975***; R. E. Butts and J. Hintikka, editors), Reidel, Dordrecht, pp. 129–145.

R. L. VAUGHT

[1986] *Alfred Tarski's work in model theory*, this JOURNAL, vol. 51, pp. 869–882.

M. YNTEMA

[1964] *A detailed argument for the Post-Linial theorems*, ***Notre Dame Journal of Formal Logic***, vol. 5, pp. 37–51.

A. P. ZAMJATIN

[1978] *A nonabelian variety of groups has an undecidable elementary theory*, ***Algebra i Logika***, vol. 17, pp. 20–27; English translation, ***Algebra and Logic***, vol. 17, pp. 13–17.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SOUTH CAROLINA 29208