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AN APPLICATION OF THE SEPARATION THEOREM FOR HERMITIAN MATRICES

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ABSTRACT. Suppose H is an $n \times n$ hermitian matrix over the complex field partitioned as $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, where C is invertible. Using the separation theorem on eigenvalues of hermitian matrices, bounds are obtained for the eigenvalues of $(H/C) = A - BC^{-1}B^*$ in terms of the eigenvalues of H and C.

I. Introduction. Suppose H is an hermitian matrix of order n partitioned as

$$H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

If C is nonsingular, the Schur complement of C in H is $A - BC^{-1}B^* = (H/C)$. Haynsworth proved in [2] that the inertia of H, denoted $\mathrm{In}(H)$, is $\mathrm{In}(H/C) + \mathrm{In}(C)$. The purpose of this paper is to determine bounds for the eigenvalues of (H/C) in terms of the eigenvalues of H and C. Our main tool will be the well-known interlacing theorem for hermitian matrices, which we now state for completeness.

Theorem [3]. *Suppose H is an $n \times n$ hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let A be the principal submatrix of H obtained by deleting the kth row and kth column of H. If $\alpha_1 \geq \cdots \geq \alpha_{n-1}$ are the eigenvalues of A, then*

$$\lambda_1 \geq \alpha_1 \geq \lambda_2 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq \lambda_n. \tag{1}$$

From this classical theorem, it follows easily that if A is a principal submatrix of H of order p with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_p$, then

$$\lambda_i \geq \alpha_i \geq \lambda_{n-p+i} \quad \text{for } i = 1, \cdots, p. \tag{2}$$

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With regard to notation, we write $H(i, k, \cdots, n \mid j, k, \cdots, n)$ to denote the minor of H with rows indexed by $(i, k, \cdots, n)$ and columns indexed by $(j, k, \cdots, n)$, where, of course, $1 \leq i, j \leq k-1$. Also, sometimes we find it convenient to denote the eigenvalues of a $p \times p$ hermitian matrix, M, by $\lambda_1(M) \geq \cdots \geq \lambda_p(M)$.

II. Bounds for the eigenvalues of (H/C). Assume $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is hermitian of order n, A is of order $k-1$, and thus C is of order $n-k+1$. Further, suppose C is invertible. Now, if we set $(H/C) = (d_{ij})$, then Crabtree and Haynsworth [1] have shown

$$(3) \qquad d_{ij} = \frac{H(i, k, \cdots, n \mid j, k, \cdots, n)}{\det(C)} \quad \text{for } 1 \leq i, j \leq k-1.$$

If we let $E = (e_{ij})$ where $e_{ij} = H(i, k, \cdots, n \mid j, k, \cdots n)$ for $1 \leq i, j \leq k-1$, then $(1/\det(C)) \cdot E = (H/C)$. It is easy to verify that E is a principal submatrix of the $(n-k+2)$-compound matrix of H, $C_{n-k+2}(H)$, which is hermitian. Then the eigenvalues of $C_{n-k+2}(H)$, say

$$\partial_1 \geq \cdots \geq \partial_{\binom{n}{n-k+2}},$$

are the $\binom{n}{n-k+2}$ products $\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_{n-k+2}}$, where $1 \leq i_1 < i_2 < \cdots < i_{n-k+2} \leq n$ [4, p. 24], where each λ_k is an eigenvalue of H.

Thus, using (2), we have

$$(4) \qquad \partial_i \geq \lambda_i(E) \geq \partial_{\binom{n}{n-k+2}-(k-1)+i} \quad \text{for } i = 1, \cdots, k-1.$$

Finally, if $\det(C) > 0$, we get

$$(5) \quad \partial_i/\det(C) \geq \lambda_i(H/C) \geq \partial_{\left(\binom{n}{n-k+2}-k+i+1\right)}/\det(C) \text{ for } i = 1, \cdots, k-1.$$

We have proved

Theorem 1. *Suppose* $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *is an hermitian matrix with the dimensions of* A *and* C *as specified earlier. Assume* $\det(C) > 0$, *and let* $C_{n-k+2}(H)$ *denote the* $(n-k+2)$*-compound matrix of* H. *If we denote the eigenvalues of* $C_{n-k+2}(H)$, C, *and* (H/C), *respectively, by*

$$\partial_1 \geq \cdots \geq \partial_{\binom{n}{n-k+2}};$$

$\alpha_1 \geq \cdots \geq \alpha_{n-k+1}$; *and* $\beta_1 \geq \cdots \geq \beta_{k-1}$, *then*

$$\frac{\partial_i}{\alpha_1 \cdots \alpha_{n-k+1}} \geq \beta_i \geq \frac{\partial_{\left(\binom{n}{n-k+2}-k+i+1\right)}}{\alpha_1 \cdots \alpha_{n-k+1}} \quad \textit{for } i = 1, \cdots, k-1.$$

Clearly, the above result holds *a fortiori* for H positive definite. In this case, $\lambda_1 \cdots \lambda_{n-k+2}$ is the largest eigenvalue of $C_{n-k+2}(H)$ and $\lambda_{k-1} \cdots \lambda_n$ is the smallest eigenvalue of $C_{n-k+2}(H)$, and we obtain a

Corollary. *Under the hypotheses of the theorem with H positive definite, then*

$$\frac{\lambda_1 \cdots \lambda_{n-k+2}}{\alpha_1 \cdots \alpha_{n-k+1}} \geq \beta_i \geq \frac{\lambda_{k-1} \cdots \lambda_n}{\alpha_1 \cdots \alpha_{n-k+1}} \quad \textit{for } i = 1, 2, \cdots, k-1.$$

We make two simple observations concerning the Corollary. For $k = 2$, the Corollary becomes

$$\frac{\lambda_1 \cdots \lambda_n}{\alpha_1 \cdots \alpha_{n-1}} \geq \det(H/C) \geq \frac{\lambda_1 \cdots \lambda_n}{\alpha_1 \cdots \alpha_{n-1}},$$

which yields $\det(H) = \det(C)\det(H/C)$, a special case of Schur's identity [2, p. 74] since C is of order $n-1$. For $k = 3$, the Corollary yields

$$\det(H/C)/\lambda_n \geq \beta_i \geq \det(H/C)/\lambda_1 \quad \text{for } i = 1, 2,$$

and thus $1/\lambda_n \geq 1/\beta_i \geq 1/\lambda_1$ for $i = 1, 2$, a reciprocal separation property. Further, we obtain $\lambda_1^2 \geq \beta_1\beta_2 \geq \lambda_n^2$ from the above inequality.

III. The positive definite case. Suppose A is an $n \times n$ positive definite matrix. Denote by A_k the principal submatrix of A contained in rows $1, 2, \cdots, k$, for $k = 1, \cdots, n-1$, and let $\lambda_n(A)$ be the minimal eigenvalue of A. As before, $\lambda_1(A)$ denotes the maximal eigenvalue of A. The following theorem and proof is similar to a result of Watford [5, Theorem 4] on M-matrices.

Theorem 2. *Suppose A is a positive definite matrix of order n. Then*

$$\lambda_n(A) \leq \lambda_n(A|A_1) \leq \cdots \leq \lambda_n(A|A_{n-1}), \tag{2.1}$$

and

$$\lambda_1(A|A_{n-1}) \leq \cdots \leq \lambda_1(A|A_1) \leq \lambda_1(A). \tag{2.2}$$

Proof. Assume, first, that A is partitioned as

$$A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}. \tag{6}$$

Now $\lambda_1(A^{-1}) = 1/\lambda_n(A)$, and since $(A|B)^{-1}$ is a principal submatrix of A^{-1} [5, p. 251], we have $\lambda_1[(A|B)^{-1}] \leq \lambda_1(A^{-1})$, using the separation theorem. Thus it follows that

$$\lambda_n(A) \leq \lambda_n(A|B). \tag{7}$$

Next, we note that if $B = A_{p+1}$ in (6), and

$$A_{p+1} = \begin{pmatrix} A_p & B_{12} \\ B_{12}^* & a_{p+1,p+1} \end{pmatrix};$$

then

$$(A|A_{p+1}) = ((A|A_p)|(A_{p+1}|A_p)),$$

the Haynsworth quotient property [1]. Using (7), we have

$$\lambda_n(A|A_{p+1}) \geq \lambda_n(A|A_p) \geq \lambda_n(A),$$

and statement (2.1) is immediate. We obtain (2.2) by noting $\lambda_n(A) = 1/\lambda_1(A)$.

IV. **Conclusion.** There exist matrices for which the bounds of Theorem 1 are exact. However, even if $H = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$, the bounds may give only rough estimates for the eigenvalues of $(H/C) = A$. For example, if $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, then $H = A \dotplus C$ has eigenvalues $\lambda_1 = 7$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\lambda_4 = 1$. But $(H/C) = A$ has eigenvalues $\beta_1 = 4$, $\beta_2 = 2$, and $C_{n-k+2}(H) = C_3(H)$ has eigenvalues $\partial_1 = 56$, $\partial_2 = 14$, $\partial_3 = 8$, The theorem yields $56/7 \geq \beta_1 \geq 14/7$ and $14/7 \geq \beta_2 \geq 8/7$. It seems likely that one could obtain "tighter" bounds in general by an application of the Courant minimax theorem for hermitian matrices—results which we shall not investigate in this paper.

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