

5-1973

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Publication Info

Proceedings of the American Mathematical Society, Volume 38, Issue 3, 1973, pages 625-631.

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ON A NONLINEAR STOCHASTIC INTEGRAL EQUATION OF THE HAMMERSTEIN TYPE

W. J. PADGETT

ABSTRACT. A nonlinear stochastic integral equation of the Hammerstein type in the form

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s)$$

is studied where $t \in S$, a σ -finite measure space with certain properties, $\omega \in \Omega$, the supporting set of a probability measure space (Ω, A, P) , and the integral is a Bochner integral. A random solution of the equation is defined to be a second order vector-valued stochastic process $x(t; \omega)$ on S which satisfies the equation almost certainly. Using certain spaces of functions, which are spaces of second order vector-valued stochastic processes on S , and fixed point theory, several theorems are proved which give conditions such that a unique random solution exists.

1. **Introduction.** The purpose of this note is to study the existence and uniqueness of a *random solution* of a nonlinear stochastic integral equation of the Hammerstein type of the form

$$(1.1) \quad x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s),$$

where

(i) S is a locally compact metric space with metric d defined on $S \times S$ and μ is a complete σ -finite measure defined on the collection of Borel subsets of S ;

(ii) $\omega \in \Omega$, where Ω is the supporting set of the probability measure space (Ω, A, P) ;

(iii) $x(t; \omega)$ is the unknown vector-valued random variable for each $t \in S$;

(iv) $h(t; \omega)$ is the stochastic free term defined for $t \in S$;

(v) $k(t, s; \omega)$ is the stochastic kernel defined for t and s in S ; and

(vi) $f(t, x)$ is a vector-valued function of $t \in S$ and x .

The integral in equation (1.1) is interpreted as a Bochner integral [12].

Received by the editors July 3, 1972 and, in revised form, August 31, 1972.

AMS (MOS) subject classifications (1970). Primary 60H20, 45G99; Secondary 60G99.

Key words and phrases. Stochastic Hammerstein integral equation, nonlinear stochastic integral equation, random solutions, second order processes.

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Further assumptions concerning the functions in (1.1) will be stated in §2.

The equation (1.1) is a generalization of stochastic integral equations studied by Padgett and Tsokos [9], Tsokos [11], and Anderson [1]. Also, equation (1.1) is a stochastic version of the deterministic integral equations which were investigated by Petryshyn and Fitzpatrick [10], Browder and Gupta [5], Browder, de Figueiredo, and Gupta [6], among others.

In order to investigate the stochastic integral equation (1.1), we will define several spaces of functions which are spaces of second order vector-valued stochastic processes on S and will use certain aspects of the "theory of admissibility" of Banach spaces as introduced into the study of integral equations by Corduneanu [7] and the methods of "probabilistic functional analysis" [3].

2. Preliminaries. We will further assume that S is the union of a countable family of compact subsets $\{C_n\}$ having the properties that $C_1 \subset C_2 \subset C_3 \subset \dots$ and that for any other compact set in S there is a C_i which contains it [2].

We define $C = C(S, L_2(\Omega, A, P))$ to be the space of all continuous functions from S into the space $L_2(\Omega, A, P)$ with the topology of uniform convergence on compacta. That is, for each fixed $t \in S$, $x(t; \omega)$ is a vector-valued random variable such that

$$\|x(t; \omega)\|_{L_2(\Omega, A, P)}^2 = \int_{\Omega} |x(t; \omega)|^2 dP(\omega) < \infty.$$

It may be noted that $C(S, L_2(\Omega, A, P))$ is a locally convex space [12, pp. 24-26] whose topology is defined by the countable family of seminorms given by

$$\|x(t; \omega)\|_n = \sup_{t \in C_n} \|x(t; \omega)\|_{L_2(\Omega, A, P)}, \quad n = 1, 2, \dots$$

Moreover, $C(S, L_2(\Omega, A, P))$ is complete relative to this topology since $L_2(\Omega, A, P)$ is complete.

We further define $BC = BC(S, L_2(\Omega, A, P))$ to be the Banach space of all bounded continuous functions from S into $L_2(\Omega, A, P)$ with norm

$$\|x(t; \omega)\|_{BC} = \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, A, P)}.$$

The space $BC \subset C$ is the space of all second order vector-valued stochastic processes defined on S which are bounded and continuous in mean-square.

We will consider the functions $h(t; \omega)$ and $f(t, x(t; \omega))$ to be in the space $C(S, L_2(\Omega, A, P))$. With respect to the stochastic kernel we assume that for each pair (t, s) , $k(t, s; \omega) \in L_{\infty}(\Omega, A, P)$ and denote the norm by

$$\| \|k(t, s; \omega)\| \| = \|k(t, s; \omega)\|_{L_{\infty}(\Omega, A, P)} = P\text{-ess sup}_{\omega \in \Omega} |k(t, s; \omega)|.$$

Also, we will suppose that $k(t, s; \omega)$ is such that

$$\| \|k(t, s; \omega)\| \|x(s; \omega)\|_{L_2(\Omega, A, P)}$$

is μ -integrable with respect to s for each $t \in S$ and $x(s; \omega)$ in $C(S, L_2(\Omega, A, P))$, and that there exists a real-valued function G defined μ -a.e. on S so that $G(s) \|x(s; \omega)\|_{L_2(\Omega, A, P)}$ is μ -integrable and, for each pair $(t, s) \in S \times S$,

$$\| \|k(t, u; \omega) - k(s, u; \omega)\| \|x(u; \omega)\|_{L_2(\Omega, A, P)} \leq G(u) \|x(u; \omega)\|_{L_2(\Omega, A, P)}$$

μ -a.e. Further, for almost all $s \in S$, $k(t, s; \omega)$ will be continuous in t from S into $L_\infty(\Omega, A, P)$.

We now define the integral operator T on $C(S, L_2(\Omega, A, P))$ by

$$(2.1) \quad (Tx)(t; \omega) = \int_S k(t, s; \omega)x(s; \omega) d\mu(s),$$

where the integral is a Bochner integral. From the conditions on $k(t, s; \omega)$, we have that for each $t \in S$, $(Tx)(t; \omega) \in L_2(\Omega, A, P)$ and that $(Tx)(t; \omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem. That is, $(Tx)(t; \omega) \in C(S, L_2(\Omega, A, P))$.

LEMMA 2.1. *The linear operator T defined by equation (2.1) is continuous from $C(S, L_2(\Omega, A, P))$ into itself.*

PROOF. Note that $C(S, L_2(\Omega, A, P))$ is a Fréchet space with metric d^* defined by the Fréchet combination of the sequence of seminorms $\|\cdot\|_n$, $n=1, 2, \dots$.

Define the sequence of linear operators $\{T_M\}$, $M=1, 2, \dots$, by

$$(T_Mx)(t; \omega) = \int_{C_M} k(t, s; \omega)x(s; \omega) d\mu(s).$$

Hence, as $M \rightarrow \infty$ we have $(T_Mx)(t; \omega) \rightarrow (Tx)(t; \omega)$.

Let $\{x_j(t; \omega)\}$ be a sequence of functions converging to $x(t; \omega)$ in $C(S, L_2(\Omega, A, P))$. Then by definition of the seminorms, for each M

$$\begin{aligned} & \| (T_Mx)(t; \omega) - (T_Mx_j)(t; \omega) \|_n \\ & \leq \sup_{t \in C_n} \int_{C_M} \| \|k(t, s; \omega)\| \|x(s; \omega) - x_j(s; \omega)\|_{L_2(\Omega, A, P)} d\mu(s). \end{aligned}$$

Since $\|x(s; \omega) - x_j(s; \omega)\|_{L_2(\Omega, A, P)} \rightarrow 0$ uniformly on the compact set C_M , for $\epsilon > 0$ there exists a positive integer N_M such that $j \geq N_M$ implies

$$\| (T_Mx)(t; \omega) - (T_Mx_j)(t; \omega) \|_n < \epsilon \sup_{t \in C_n} \int_{C_M} \| \|k(t, s; \omega)\| d\mu(s).$$

Now, by the conditions on $k(t, s; \omega)$, there exists a constant K_n such that $\|k(t, s; \omega)\| \leq K_n$ for all $t \in C_n$ and almost all s . Hence, for $j \geq N_M$

$$\|(T_M x)(t; \omega) - (T_M x_j)(t; \omega)\|_n < \varepsilon K_n \mu(C_M).$$

Since convergence in every seminorm is equivalent to convergence in the metric d^* , $(T_M x_j)(t; \omega)$ converges to $(T_M x)(t; \omega)$ in $C(S, L_2(\Omega, A, P))$ for each M . Therefore, by [8, p. 54], T is continuous from $C(S, L_2(\Omega, A, P))$ into itself.

Let B and D be Banach spaces. The pair (B, D) is said to be *admissible* with respect to a linear operator T if $T(B) \subset D$.

LEMMA 2.2. *If T is a continuous linear operator from $C(S, L_2(\Omega, A, P))$ into itself and $B, D \subset C(S, L_2(\Omega, A, P))$ are Banach spaces stronger than $C(S, L_2(\Omega, A, P))$ such that (B, D) is admissible with respect to T , then T is continuous from B into D .*

The lemma follows from the closed-graph theorem.

From Lemmas 2.1 and 2.2 it follows that T defined by equation (2.1) is a bounded linear operator from B into D .

By a *random solution* of the equation (1.1) we will mean a function $x(t; \omega)$ in $C(S, L_2(\Omega, A, P))$ which satisfies the equation P -a.e.

3. Existence of a random solution. We now present theorems concerning the existence and uniqueness of a random solution of the equation (1.1).

THEOREM 3.1. *We consider the stochastic integral equation (1.1) subject to the following conditions:*

(i) *B and D are Banach spaces stronger than $C(S, L_2(\Omega, A, P))$ such that (B, D) is admissible with respect to the integral operator defined by equation (2.1);*

(ii) *$x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator from the set*

$$Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$$

into the space B satisfying the Lipschitz condition

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \lambda \|x(t; \omega) - y(t; \omega)\|_D$$

for $x(t; \omega), y(t; \omega) \in Q(\rho)$, where ρ and λ are constants;

(iii) *$h(t; \omega) \in D$.*

Then there exists a unique random solution of (1.1) in $Q(\rho)$, provided $\lambda K < 1$ and

$$\|h(t; \omega)\|_D + K \|f(t, 0)\|_B \leq \rho(1 - \lambda K),$$

where K is the norm of T .

PROOF. Define the operator U from $Q(\rho)$ into D by

$$(Ux)(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s).$$

Then from the conditions of the theorem

$$\begin{aligned} \|(Ux)(t; \omega)\|_D &\leq \|h(t; \omega)\|_D + K \|f(t, x(t; \omega))\|_B \\ &\leq \|h(t; \omega)\|_D + K \|f(t, 0)\|_B + K\lambda \|x(t; \omega)\|_D \leq \rho. \end{aligned}$$

Hence, $(Ux)(t; \omega) \in Q(\rho)$.

Now, for $x(t; \omega), y(t; \omega) \in Q(\rho)$ we have by condition (ii) that

$$\begin{aligned} \|(Ux)(t; \omega) - (Uy)(t; \omega)\|_D &= \left\| \int_S k(t, s; \omega) [f(s, x(s; \omega)) - f(s, y(s; \omega))] d\mu(s) \right\|_D \\ &\leq K \|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \\ &\leq \lambda K \|x(t; \omega) - y(t; \omega)\|_D. \end{aligned}$$

Since $\lambda K < 1$, U is a contraction on $Q(\rho)$.

Therefore, by Banach's fixed point theorem there exists a unique $x^*(t; \omega) \in Q(\rho)$ which is a fixed point of U , that is, $x^*(t; \omega)$ is the unique random solution of equation (1.1).

A similar theorem may be obtained when f is a nonlinear contraction on $Q(\rho)$ [4].

THEOREM 3.2. *Assume that the stochastic integral equation (1.1) satisfies the following conditions:*

- (i) same as Theorem 3.1(i);
- (ii) $x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator from the set $Q(\rho)$ into the space B satisfying

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \phi(\|x(t; \omega) - y(t; \omega)\|_D)$$

for $x(t; \omega), y(t; \omega) \in Q(\rho)$, where ϕ is a real-valued continuous function such that $\phi(s) < s$ for $s > 0$;

- (iii) $h(t; \omega) \in D$.

Then there exists a unique random solution of (1.1) in $Q(\rho)$, provided $K \leq 1$ and $\|h(t; \omega)\|_D + K \|f(t, 0)\|_B \leq \rho(1 - K)$, where K is the norm of T .

The proof of Theorem 3.2 is similar to that of Theorem 3.1 except that the fixed point theorem of Boyd and Wong [4] is used.

The following is a useful application of Theorem 3.1.

COROLLARY 3.1. *Suppose the stochastic integral equation (1.1) satisfies the following conditions:*

(i) $\sup_{t \in S} \int_S \|k(t, s; \omega)\| d\mu(s) < \infty$;

(ii) $f(t, x)$ is a continuous function of $t \in S$ uniformly in x such that for $\|x(t; \omega)\|_{BC}, \|y(t; \omega)\|_{BC} \leq \rho$

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_{L_2(\Omega, A, P)} \leq \lambda \|x(t; \omega) - y(t; \omega)\|_{L_2(\Omega, A, P)}$$

for each $t \in S$, where λ and ρ are constants;

(iii) $h(t; \omega)$ is a bounded continuous function from S into $L_2(\Omega, A, P)$.

Then there exists a unique random solution of equation (1.1), provided $\sup_{t \in S} \int_S \|k(t, s; \omega)\| d\mu(s)$, λ , and $\|f(t, 0)\|_{BC}$ are sufficiently small.

PROOF. We must show that condition (i) implies that the pair (BC, BC) is admissible with respect to the integral operator T defined by equation (2.1). Let $x(t; \omega) \in BC(S, L_2(\Omega, A, P))$. Then by the properties of the Bochner integral

$$\begin{aligned} \|(Tx)(t; \omega)\|_{L_2(\Omega, A, P)} &\leq \int_S \|k(t, s; \omega)x(s; \omega)\|_{L_2(\Omega, A, P)} d\mu(s) \\ &\leq \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, A, P)} \int_S \|k(t, s; \omega)\| d\mu(s) \\ &\leq \|x(t; \omega)\|_{BC} \sup_{t \in S} \int_S \|k(t, s; \omega)\| d\mu(s). \end{aligned}$$

Hence, $(Tx)(t; \omega) \in BC(S, L_2(\Omega, A, P))$, that is, (BC, BC) is admissible with respect to T .

Conditions (ii)–(iii) clearly imply that conditions (ii)–(iii) of Theorem 3.1 hold. Thus, by Theorem 3.1, there exists a unique random solution of equation (1.1).

Other corollaries of Theorems 3.1 and 3.2 may be obtained by choosing different Banach spaces contained in the space $C(S, L_2(\Omega, A, P))$ and different conditions on f and k .

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