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A NOTE ON ROE'S CHARACTERIZATION OF THE SINE FUNCTION

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ABSTRACT. Let $f^{(n)} n = 0, \pm 1, \pm 2, \dots$ be a sequence of complex valued functions on the real line with $(d/dx)f^{(n)} = f^{(n+1)}$ and satisfying inequalities $|f^{(n)}(x)| \le M_n(1+|x|)^k$ where as $n \to \infty$ the growth conditions $\underline{\lim}\, M_n(1+\varepsilon)^{-n} = 0$ and $\underline{\lim}\, M_{-n}(1+\varepsilon)^{-n} = 0$ hold for all $\varepsilon > 0$. Then $f^{(0)}(x) = p(x)e^{ix} + q(x)e^{-ix}$ where p and q are polynomials of degree at most k.

In his paper [1] J. Roe proves

Theorem (Roe [1]). *Let $\{f^{(n)}\}_{n=-\infty}^{\infty}$ be a two way infinite sequence of real valued functions defined on the real line R. Assume $f^{(n+1)}(x) = (d/dx)f^{(n)}(x)$ and that there is a constant M so that $|f^{(n)}(x)| \le M$ for all n and x. Then $f^{(0)}(x) = a\sin(x+\phi)$ for some real constants a and ϕ.*

This gives a rather striking characterization of the sine functions $a\sin(x+\phi)$ in terms of the size of their derivatives and antiderivatives. In this note we show that the bounds $|f^{(n)}(x)| \le M$ can be relaxed to $|f^{(n)}(x)| \le M_n(1+|x|)^{\alpha}$ with $0 \le \alpha < 1$ and where the constants only need to have supexponential growth. More generally:

Theorem. *Let $\{f^{(n)}\}_{n=-\infty}^{\infty}$ be a sequence of complex valued functions defined on the real numbers with*

$$f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x) \tag{1}$$

and so that there are constants $M_n \ge 0$, $\alpha \in [0,1)$, and a nonnegative integer k satisfying

$$|f^{(n)}(x)| \le M_n(1+|x|)^{k+\alpha}. \tag{2}$$

If

$$\underline{\lim}_{n\to\infty} \frac{M_n}{(1+\varepsilon)^n} = 0 \quad \textit{all } \varepsilon > 0 \tag{3}$$

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and

(4) $$\lim_{n\to\infty} \frac{M_{-n}}{(1+\varepsilon)^n} = 0 \quad \textit{all } \varepsilon > 0,$$

then

(5) $$f^{(0)}(x) = p(x)e^{ix} + q(x)e^{-ix}$$

where $p(x)$ *and* $q(x)$ *are polynomials of degree at most* k.

Remark 1. The conclusion of the theorem can be sharpened by giving a more precise description of the functions $f^{(n)}$. If $k = 0, 1, 2, \ldots$ and $n = 0, \pm 1, +2, \ldots$ then define

$$\begin{aligned} p_{n,k}(x) &= (1 + \frac{d}{dx})^n x^k \\ &= \sum_{m=0}^{k} \frac{n(n-1)\cdots(n+m-k+1)}{(k-m)!} k(k-1)\cdots(m+1)x^m \end{aligned}$$

where for negative n we expand $(1 + d/dx)^n$ formally by use of Taylor's theorem. Then $p_{n,k}(x)$ is a polynomial of degree k and

$$p_{n,k}(x) + p'_{n,k}(x) = \left(1 + \frac{d}{dx}\right) p_{n,k}(x) = p_{n+1,k}(x).$$

When $n \geq 0$ these are Laguerre polynomials. This is because

$$(-1)^k p_{n,k}(-x) = \left(1 - \frac{d}{dx}\right)^n x^k = e^x \frac{d^n}{dx^n}(x^k e^{-x}).$$

See for example [0, p. 204]. This last equation implies that if λ is a complex number and

$$f_{k,\lambda}^{(n)}(x) = \lambda^n p_{n,k}(\lambda x)e^{\lambda x}$$

then $\{f_{k,\lambda}^{(n)}\}_{n=-\infty}^{\infty}$ satisfies equation (1). Then if $\{f^{(n)}\}_{n=-\infty}^{\infty}$ is a sequence of functions satisfying the hypothesis of the theorem then there are complex numbers $a_0, \ldots, a_k, b_0, \ldots, b_k$ so that

$$f^{(n)} = \sum_{m=0}^{k} (a_m f_{m,i}^{(n)} + b_m f_{m,-i}^{(n)})$$

where $(i)^2 = -1$. The proof of this from the theorem is done by induction on k. The details are left to the reader.

Remark 2. The functions $f_{k,\lambda}^{(n)}(x)$ just defined satisfy

$$|f_{k,\lambda}^{(n)}(x)| \leq (k+1)! n^k \max(|\lambda|^n, |\lambda|^{n+k})(1+|x|)^k e^{\mathrm{Re}(\lambda)} x.$$

By giving λ pure imaginary values close to i or $-i$ we see that there is no obvious weakening of the growth conditions (2), (3), or (4).

Remark 3. It is impossible to replace the interval $(-\infty, \infty)$ by a half infinite interval. The functions $f^{(n)}(x) = (-1)^n e^{-x}$ on $(0, \infty)$ yield a counterexample. (This observation is due to David Richman.)

Proof of the theorem. Let $f(x) = f^{(0)}(x)$. The, following [1], we will show the support of the Fourier transform $\hat{f}$ of f contained in the set $\{1, -1\}$. As the integral defining the Fourier transform may diverge, we define it as a distribution, that is as a linear functional on the vector spaces $\mathscr{S}$ of rapidly decreasing functions on R. Explicitly the value of $\hat{f}$ on $\phi \in \mathscr{S}$ is

$$\langle \hat{f}, \phi \rangle = \langle B, \hat{\phi} \rangle = \int_{-\infty}^{\infty} f(x)\hat{\phi}(x)\, dx.$$

Here we follow the notation of [2, Chapter 7].

Suppose it has been shown that the support of $\hat{f}$ is contained in $\{1, -1\}$. Then a standard result [2, Theorem 6.25, p. 150] implies there is an $m \geq 0$ and complex numbers a_j, b_j $0 \leq j \leq m$ so that

$$\hat{f} = \sum_{j=0}^{m} a_j \delta_1^{(j)} + \sum_{j=0}^{m} b_j \delta_{-1}^{(j)}$$

where δ_1 (resp. δ_{-1}) is the delta function at 1 (resp. at -1) and $\delta_1^{(j)}$ is the jth distributional derivative of δ_1. This Fourier transform can be inverted to give that $f(x) = f^{(0)}(x)$ has the form given by (5) with $p(x)$ and $g(x)$ polynomials (of degree at most m). But $|f(x)| \leq |f^{(0)}(x)| \leq M_0(1 + |x|)^{k+\alpha}$. This implies the polynomials have degree at most k.

This reduces the proof to showing

Lemma 1. *The conditions* (2) *and and* (3) *imply the support of* $\hat{f}$ *is disjoint from* $(1, \infty)$ *and* $(-\infty, -1)$.

Lemma 2. *The conditions* (2) *and* (4) *imply the support of* $\hat{f}$ *is disjoint from* $(-1, 1)$.

Proof of Lemma 1. We will only show the support of $\hat{f}$ is disjoint from $(1, \infty)$, the proof for $(-\infty, -1)$ being identical. Let ϕ be a smooth function with its support, $\operatorname{spt}(\phi)$, in $(1, \infty)$. Then $\operatorname{spt}(\phi) \subseteq [r, \infty)$ for some $r > 1$. We now need to show $\langle \hat{f}, \phi \rangle = 0$. Let $n \geq 0$ and

$$\psi_n(t) = \frac{\phi(t)}{(-it)^n}. \tag{6}$$

This is smooth as $\phi = 0$ near $t = 0$. Thus differentiating under the integral gives

$$\hat{\psi}_n^{(n)}(x) = \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{\phi(t)}{(-it)^n} e^{-itx}\, dx = \hat{\phi}(x).$$

So

$$\begin{aligned}\langle \hat{f}, \phi\rangle &= \langle f, \hat{\phi}\rangle \\ &= \langle f, \hat{\psi}_n^{(n)}\rangle \\ &= (-1)^n \langle f^{(n)}, \hat{\psi}_n\rangle \\ &= (-1)^n \int_{-\infty}^{\infty} f^{(n)}(x)\hat{\psi}_n(x)\, dx.\end{aligned}$$

By (2) this implies

$$|\langle \hat{f}, \phi\rangle| \le M_n \int_{-\infty}^{\infty} (1+|x|^{k+\alpha}|\hat{\psi}_n(x)|)\, dx. \tag{7}$$

We now estimate $|\hat{\psi}_n(x)|$. First using that $\operatorname{spt}(\phi) \subseteq [r, \infty)$,

$$|\hat{\psi}_n(x) \le \int_r^{\infty} \frac{|\phi(t)|}{t^n}\, dt \le \frac{1}{r^n}\|\phi\|_{L^1}. \tag{8}$$

Also if $x \neq 0$ then integration by parts $(k+2)$ times yields

$$\begin{aligned} |\hat{\psi}_n(x)| &= \left| \int_r^{\infty} \left(\frac{d^{k+2}}{dt^{k+2}} \frac{\phi(t)}{(-it)^n} \right) \frac{e^{-ixt}}{(-ix)^{k+2}}\, dt \right| \\ &\le \frac{1}{|x|^{k+2}} \int_r^{\infty} \left| \frac{d^{k+2}}{dt^{k+2}} \frac{\phi(t)}{t^n} \right| dt \\ &\le \frac{1}{|x|^{k+2}} \sum_{j=0}^{k+2} \binom{k+2}{j} n(n+1) \\ &\qquad \cdots (n+j-1)\|\phi^{(k+2-j}\|_{\infty} \int_r^{\infty} \frac{dt}{t^{n+j}} \\ &= \frac{1}{|x|^{k+2}} \sum_{j=0}^{k+2} \binom{k+2}{j} \frac{n(n+1)\cdots(n+j-1)\|\phi^{(k+2-j)}\|}{(n+j-1)r^{n+j-1}} \\ &\le \frac{c_1(k, \phi)n^{k+1}}{|x|^{k+2}r^{n-1}} \end{aligned} \tag{9}$$

where $c_1(k, \phi)$ is a constant depending only on k and ϕ. This can be combined with (8) to give

$$|\psi_n(x)| \le \begin{cases} 1/r^n \|\phi\|_{L^1} & |x| \le 1, \\ \frac{c_1(k, \phi)n^{k+1}}{|k|^{k+2}r^{n-1}} & |x| > 1. \end{cases} \tag{10}$$

Using this in (7) gives an estimate of the form

$$|\langle \hat{f}, \phi\rangle| \le c_2(k, \alpha, \phi)\frac{n^{k+1}}{r^{n-1}} M_n. \tag{11}$$

Let ε be so that $1 < 1+\varepsilon < r$. Then for large n

$$\frac{n^{k+1}}{r^{n-1}} < \frac{1}{(1+\varepsilon)^n}.$$

Using this and the condition (3) and (11)

$$|\langle \hat{f}, \phi\rangle| \le c_2(k, \alpha, \phi) \varliminf_{n\to\infty} \frac{M_n}{(1+\varepsilon)^n} = 0.$$

Therefore $\langle \hat{f}, \phi\rangle = 0$ for all ϕ with support in $(1, \infty)$, i.e., the support of $\hat{f}$ is disjoint from $(1, \infty)$ as claimed.

Proof of Lemma 2. The proof is very similar to the proof of Lemma 1. Let ϕ be a smooth function with support in $(-1, 1)$. Then for some $r < 1$ the inclusion $\operatorname{spt}(\phi) \subseteq [-r, r]$ holds.

$$\begin{aligned}\langle \hat{f}, \phi\rangle &= \langle f^{(0)}, \hat{\phi}\rangle = \langle \frac{d^n}{dx^n} f^{(-n)}, \hat{\phi}\rangle \\ &= (-1)^n \langle f^{(-n)}, \hat{\phi}^{(n)}\rangle \\ &= (-1)^n \int_{-\infty}^{\infty} f^{(-n)}(x)\hat{\phi}^{(n)}(x)\,dx.\end{aligned}$$

By (2) this implies

$$(12) \qquad |\langle \hat{f}, \phi\rangle| \le M_{-n} \int_{-\infty}^{\infty} (1+|x|)^{k+\alpha} |\hat{\phi}^{(n)}(x)|\,dx.$$

Differenting under the integral and using $\operatorname{spt}(\phi) \subseteq [-r, r]$

$$(13) \qquad \hat{\phi}^{(n)}(x) = \int_{-r}^{r} (-it)^n \phi(t) e^{-itx}\,dt.$$

Thus

$$(14) \qquad |\hat{\phi}^{(n)}(x)| \le \int_{-r}^{r} |t|^n |\phi(t)|\,dt \le 2r^n \|\phi\|_{L^1}.$$

Also for $x \ne 0$, integration by parts $(k+2)$ times and calculations similar to those of inequality (9) yield

$$(15) \qquad \begin{aligned}|\hat{\phi}^{(n)}(x)| &= \left| \int_{-r}^{r} \left(\frac{d^{k+2}}{dt^{k+2}} (-it)^n \phi(t) \right) \frac{e^{-itx}}{(-ix)^{k+2}}\,dt \right| \\ &\le \frac{1}{|x|^{k+2}} \int_{-r}^{r} \left| \frac{d^{k+2}}{dt^{k+2}} (t^n \phi(t)) \right| dt \\ &\le \frac{c_3(k, \phi) h^{k+2}}{|x|^{k+2}} r^{n-k-2}.\end{aligned}$$

Putting the last two estimates together

$$(16) \qquad |\hat{\phi}^{(n)}(x)| \le \begin{cases} 2r^n \|\phi\|_{L^1} & |x| \le 1, \\ \frac{c_3(k, \phi) n^{k+2}}{|x|^{k+2}} r^{n-k-2} & |x| > 1. \end{cases}$$

Putting this in (12) gives an estimate

$$|\langle \hat{f}, \phi\rangle| \le c_4(k, \alpha, \phi) M_{-n} n^{k+2} r^{n-k-2}.$$

The proof is now completed in the same manner as the proof of Lemma 1.

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