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On stationary and moving interface cracks with frictionless contact in anisotropic bimaterials

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The asymptotic structure of near-tip fields around stationary and steadily growing interface cracks, with frictionless crack surface contact, and in anisotropic bimaterials, is analysed with the method of analytic continuation, and a complete representation of the asymptotic fields is obtained in terms of arbitrary entire functions. It is shown that when the symmetry, if any, and orientation of the anisotropic bimaterial is such that the in-plane and out-of-plane deformations can be separated from each other, the in-plane crack-tip fields will have a non-oscillatory, inverse-squared-root type stress singularity, with angular variations clearly resembling those for a classical mode II problem when the bimaterial is orthotropic. However, when the two types of deformations are not separable, it is found that an oscillatory singularity different than that of the counterpart open-crack problem may exist at the crack tip for the now coupled in-plane and out-of-plane deformation. In general, a substantial part of the non-singular higher-order terms of the crack-tip fields will have forms that are identical to those for the counterpart open-crack problem, which give rise to fully continuous displacement components and zero tractions along the crack surfaces as well as the material interface.

1. Introduction

The asymptotic structure of interfacial, two-dimensional elastic crack-tip fields in an *anistropic* bimaterial, assuming traction-free crack surfaces (the *open-crack* assumption), is well understood today (see, for example, recent general treatments by Suo (1990), Wu (1991), Yang *et al.* (1991) and Deng (1993*a*), among others). The oscillatory singularity of the stress and deformation fields at the crack tip has the same form as that for a stationary crack along the interface of two dissimilar *isotropic* materials, which was first obtained by Williams (1959). Accordingly, the mathematically inconsistent feature, as formally demonstrated by England (1965), of crack surface interpenetration of the open-crack solution (obtained under conditions of no crack surface contact) for interface cracks in isotropic bimaterials, will also pertain to the counterpart case of anisotropic bimaterials. Fortunately, when the zone with crack surface overlapping, the *contact zone*, is sufficiently small compared to a characteristic crack length, the aforementioned undesirable feature can be alleviated with the concept of *small-scale contact* (Rice 1988), and the open-crack fields will still adequately describe the state of stress and deformation around the crack tip. However, when the contact zone is not confined to a small region near the crack tip, the notion of small-scale contact will break down and asymptotic crack-tip fields which account for the effect of crack surface contact must be used for fracture analyses.

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Solutions of some particular boundary-value problems involving stationary interfacial cracks, such as those assuming traction-free crack surfaces (see England 1965; Willis 1972), or those allowing frictionless contact between the crack surfaces over certain unknown region behind the crack tip (see Comninou 1977*a*; Comninou & Schmueser 1979; Gautesen & Dundurs 1987, 1988), all indicate that the size of the contact zone is negligible when loading is tensile but becomes significant when loading contains proper, sufficient amount of shear. In general, the contact zone size can be estimated from dimensional analyses based on the asymptotic singular crack-tip fields for the open interface crack (see Rice 1988; Zywicz & Parks 1990), and its value will depend on the type of loading (e.g. in terms of the phase angle of the complex stress intensity factor), on the amount of mismatch between the two dissimilar materials (in terms of the *oscillation index* ϵ, a non-dimensional mismatch parameter), and on the speed of crack propagation v ($v = 0$ for stationary cracks). While for a stationary crack the effect of mismatch on the size of the contact zone is often small, since the magnitude of ϵ is within the small range of 0.01 and 0.04 (Hutchinson *et al.* 1987), it might become important for a propagating crack as its speed v approaches c_{R}, the smaller of the Rayleigh wave speeds of the two component materials, since ϵ tends to infinity as v tends to c_{R} (Willis 1971; Atkinson 1977; Yang *et al.* 1991). In conclusion, when v is sufficiently close to c_{R} or when loading is mostly shear (along a proper direction), the appropriate asymptotic solutions for the crack-tip fields should be those that take into consideration of crack surface contact.

The investigation of asymptotic, interfacial crack-tip fields with crack-surface contact was pioneered by Comninou (1977*a*, *b*) for a stationary crack in a dissimilar isotropic media. Her solution for the case of frictionless crack-surface contact showed that the crack-tip stress field has the usual inverse-square-root singularity, and its angular variation resembles that for a classical mode II problem. This solution was the basis for two recent boundary-layer, elastic–plastic studies of frictionless contact at interfacial crack tips (Zywicz & Parks 1990; Aravas & Sharma 1991). More recently, a Comninou-type analysis was performed by the current author (Deng 1993*b*) for the case of a propagating interface crack in a linearly elastic, isotropic bimaterial, and a complete representation of the crack-tip stress and deformation fields was obtained in terms of several arbitrary entire functions, which can be put in the form of a Williams-type series expansion. As in the case of a stationary crack, the singular crack-tip fields resemble those for a classical mode II problem.

In this paper, we are interested in the structure of the asymptotic, elastic crack-tip fields for an interface crack, stationary or steadily moving, with frictionless contact and in a generally anisotropic bimaterial. Combined in-plane and out-of-plane deformation is considered, and a complete representation of the crack-tip stress and displacement fields is sought with the method of analytic continuation. Special attention is given to the case where the dissimilar anisotropic materials have one plane of symmetry which is so aligned that the combined deformation can be uncoupled, which includes such common dissimilar material pairs as orthotropic and transversely isotropic bimaterials. It is noted that the theory presented here can also be applied, with minor modifications, to situations where constant friction exists between the sliding crack surfaces, as in the case of fibre pullout in composites under certain idealized conditions.

2. Problem formulation

For the sake of two-dimensional asymptotic analysis, we can expand the field of view near the crack tip and consider a straight, semi-infinite crack in an otherwise infinite plane. The crack can be either moving steadily, say from left to right, with (constant) speed v or stationary ($v = 0$), and its surfaces are in frictionless contact. A right-hand rectangular cartesian coordinate system is attached to the crack tip, with its negative x-axis coinciding with the crack and the positive y-axis pointing upwards. The out-of-plane coordinate will be denoted by x_3. Let dissimilar materials 1 and 2 occupy the half-planes above and below the x-axis, respectively. Each material is itself homogeneous, linearly elastic, and anisotropic in general. As a convention, explicit single subscripts 1 or 2 will be used to mark quantities associated with materials 1 or 2 whenever confusion may arise, but generic notations will also be used for those that apply to both materials, although they may take different values due to material dissimilarity.

According to the Stroh formalism (Stroh 1962), the stress and displacement fields for each half-plane of the present problem can be expressed in terms of three arbitrary analytic functions $f_1(z_1), f_2(z_2)$, and $f_3(z_3)$, where $z_i = x + p_i\, y\, (i = 1, 2, 3)$, as

$$\boldsymbol{u} = 2\,\mathrm{Re}\,[\boldsymbol{A}\boldsymbol{f}(z)], \quad \boldsymbol{t} = 2\,\mathrm{Re}\,[\boldsymbol{L}\boldsymbol{f}'(z)], \quad \boldsymbol{s} = -2\,\mathrm{Re}\,[\boldsymbol{L}\boldsymbol{\Gamma}\boldsymbol{f}'(z) - \rho v^2 \boldsymbol{A}\boldsymbol{f}'(z)], \tag{1}$$

which hold for both coupled and uncoupled in-plane and out-of-plane deformations, where vector $\boldsymbol{f}(z)$ has components f_i with the understanding that z is to be replaced for f_i by z_i, and $\boldsymbol{f}'(z)$ is formed by the derivative of f_i with respect to its argument (this convention also is used later). The symbol Re [ ] denotes the real part of the content in [ ], $\boldsymbol{u}$ (with components u_x, u_y and u_3) is the displacement vector, and $\boldsymbol{t}$ (with components σ_{yx}, σ_{yy} and σ_{y3}) and $\boldsymbol{s}$ (with components σ_{xx}, σ_{xy} and σ_{x3}) are the stress vectors (symmetry of the stress components is observed). Matrices $\boldsymbol{A}$, $\boldsymbol{L}$ and $\boldsymbol{\Gamma}$ are related to an eigenvalue problem for the material and depend on the material's fourth-order stiffness tensor $\boldsymbol{C}$, mass density ρ and crack speed v, which is assumed to be less than c_{R}, the smaller of the Rayleigh wave speeds of the two component materials. The eigenvalues $p_i\, (i = 1, 2, 3)$ are those with positive imaginary parts. For brevity, the detail of the above eigenvalue problem is not cited here but can be found in references with similar notations, for example, in Deng (1993*a*) and Yang *et al.* (1991). To simplify further derivations in the subsequent sections, we will adopt two notations introduced by Suo (1990), namely $\boldsymbol{h}(z)$, a vector of analytic functions, and $\boldsymbol{B}$, a positive definite hermitian matrix, which are

$$\boldsymbol{h}(z) = \boldsymbol{L}\boldsymbol{f}(z), \quad \boldsymbol{B} = \mathrm{i}\boldsymbol{A}\boldsymbol{L}^{-1}, \tag{2}$$

where $\mathrm{i} = \sqrt{-1}$. Hence, once a solution for $\boldsymbol{h}(z)$ is obtained, $\boldsymbol{f}(z)$ can be solved from (2), which then can be substituted into (1), with the argument z replaced by z_i for the ith component, f_i, of the vector $\boldsymbol{f}(z)$, to obtain the final solution for the stresses and displacements.

In the next section, the form of the vector of complex functions, $\boldsymbol{h}(z)$, and hence that of $\boldsymbol{f}(z)$, will be determined from the following boundary conditions. Along the positive x-axis, tractions and displacements must be continuous across the material interface:

$$\left.\begin{aligned} (\sigma_{yx})_1 &= (\sigma_{yx})_2, \quad (\sigma_{yy})_1 = (\sigma_{yy})_2, \quad (\sigma_{y3})_1 = (\sigma_{y3})_2, \\ (u_x)_1 &= (u_x)_2, \quad\;\; (u_y)_1 = (u_y)_2, \quad\;\; (u_3)_1 = (u_3)_2. \end{aligned}\right\} \tag{3}$$

And along the negative x-axis, the crack surfaces must be in frictionless contact:

$$(\sigma_{yy})_1 = (\sigma_{yy})_2 \leqslant 0, \quad (\sigma_{yx})_1 = (\sigma_{yx})_2 = 0, \quad (\sigma_{y3})_1 = (\sigma_{y3})_2 = 0, \quad (u_y)_1 = (u_y)_2. \tag{4}$$

It is reminded here that subscripts 1 and 2 signify quantities for materials 1 and 2 respectively, and subscript 3 refers to the out-of-plane direction.

3. General solution

For convenience, we will treat in the sequel row and column vectors as equivalent. It can be shown that the boundary conditions (3) and (4) are equivalent to the following:

$$\left.\begin{array}{c} \boldsymbol{t}_1 = \boldsymbol{t}_2 \quad \text{for} \quad y = 0, \quad -\infty < x < \infty, \\ \boldsymbol{u}_1 = \boldsymbol{u}_2 \quad \text{for} \quad y = 0, \quad 0 < x < \infty, \\ \boldsymbol{e}\cdot\boldsymbol{t}_1 = 0, \quad \boldsymbol{e}^*\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2) = 0, \quad \boldsymbol{e}_3\cdot\boldsymbol{t}_1 = 0 \quad \text{for} \quad y = 0, \quad -\infty < x < 0, \end{array}\right\} \tag{5}$$

where $\cdot$ implies inner product between two vectors, and unit vectors $\boldsymbol{e} = (1,0,0)$, $\boldsymbol{e}^* = (0,1,0)$ and $\boldsymbol{e}_3 = (0,0,1)$ are, respectively, the base vectors along x, y and x_3 axes. In what follows, we will apply the standard method of analytic continuation to obtain general descriptions of the stress and displacement fields in terms of arbitrary entire functions, and discuss the structure of the singularity at the crack tip. To be concise, we will omit most intermediate steps in the derivation, and only list the key ones here.

To start with, it can be shown from the first equation of (5) that

$$\boldsymbol{h}_1'(z) - \bar{\boldsymbol{h}}_2'(z) = \boldsymbol{h}_2'(z) - \bar{\boldsymbol{h}}_1'(z) = \boldsymbol{g}(z), \tag{6}$$

which holds on the whole plane, where vector $\boldsymbol{g}(z)$ is composed of entire functions and satisfies $\boldsymbol{g}(z) = -\bar{\boldsymbol{g}}(z)$ (here $\bar{\boldsymbol{g}}(z)$ denotes the complex conjugate of $\boldsymbol{g}(\bar{z})$; this convention also is used for other functions). Note that a bar over a quantity denotes the complex conjugate of that quantity. Now from the second equation of (5), we can derive

$$\boldsymbol{B}_1 \boldsymbol{h}_1'(z) + \bar{\boldsymbol{B}}_2 \bar{\boldsymbol{h}}_2'(z) = \boldsymbol{B}_2 \boldsymbol{h}_2'(z) + \bar{\boldsymbol{B}}_1 \bar{\boldsymbol{h}}_1'(z), \tag{7}$$

which is true everywhere except along the negative x-axis. It is noted here that relations in (6) and (7) can be used to express $\bar{\boldsymbol{h}}_1'(z)$, $\boldsymbol{h}_2'(z)$, and $\bar{\boldsymbol{h}}_2'(z)$ in terms of $\boldsymbol{h}_1'(z)$ and $\boldsymbol{g}(z)$. Before we go for the rest equations of (5), it is convenient to introduce two hermitian matrices, $\boldsymbol{H}$ and $\boldsymbol{G}$, which are given by

$$\boldsymbol{H} = \boldsymbol{B}_1 + \bar{\boldsymbol{B}}_2, \quad \boldsymbol{G} = \bar{\boldsymbol{B}}_1 - \bar{\boldsymbol{B}}_2. \tag{8}$$

It can be shown that when the crack speed is less than c_{R}, the smaller of the Rayleigh wave speeds of the bimaterial, $\boldsymbol{H}$ is positive definite. Finally, from the remaining requirements in (5), and making use of (6) and (7) and relations derived from them, we get, for $y = 0$, $-\infty < x < 0$,

$$\boldsymbol{U}\boldsymbol{h}_1'^{+} + \boldsymbol{V}\boldsymbol{h}_1'^{-} = \boldsymbol{W}\boldsymbol{g}(x), \tag{9}$$

which is a non-homogeneous Hilbert problem, where superscripts $+$ and $-$ imply that the negative x-axis is approached, respectively, from above and below, and matrices $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ are related to $\boldsymbol{H}$ and $\boldsymbol{G}$, and their components are given by

$$\left.\begin{array}{c} U_{1j} = |\boldsymbol{H}|\delta_{1j}, \quad U_{2j} = H_{2j}, \quad U_{3j} = |\boldsymbol{H}|\delta_{3j}, \\ V_{1j} = \Sigma\,(-1)^{k+1} M_{1k} H_{kj}, \quad V_{2j} = -U_{2j}, \quad V_{3j} = \Sigma\,(-1)^{k+1} M_{3k} H_{kj}, \\ W_{1j} = U_{1j} + \Sigma\,(-1)^{k} M_{1k} G_{kj}, \quad W_{2j} = 0, \quad W_{3j} = U_{3j} + \Sigma\,(-1)^{k} M_{3k} G_{kj}, \end{array}\right\} \tag{10}$$

where Latin indices have values 1, 2 and 3; the summations are over index k from 1 to 3; $|\boldsymbol{H}|$ stands for the determinant of $\boldsymbol{H}$; and M_{ik} denotes the minor of $\boldsymbol{H}$

associated with component H_{ik}, and is equal to the determinant of $\boldsymbol{H}$ with its ith row and kth column deleted. It is observed that in each of the matrices $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$, the dimension of the second row is different from those of the other two rows. However, since this fact will not affect the outcome of the derivation we have not attempted to modify that, for example, by multiplying every component of the second row with the product $H_{11}H_{22}$.

A solution for (9) is composed of two parts: a particular part and a homogeneous part. The simplest choice for the particular solution of (9) is a vector of entire functions that satisfies:

$$(\boldsymbol{U}+\boldsymbol{V})\,\boldsymbol{h}_1'(z)=\boldsymbol{W}\boldsymbol{g}(z). \tag{11}$$

Substitution of (10) into (11) reveals that $(\boldsymbol{U}+\boldsymbol{V})_{2j}=0$, along with $W_{2j}=0$, which implies that (11) has an infinite number of solutions. To seek a solution with a simple format, we carefully examined the relations between matrices $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$, and matrices $\boldsymbol{H}$ and $\boldsymbol{G}$, and found that the unique, particular solution of the next equation is such a solution for (11):

$$(\boldsymbol{H}+\bar{\boldsymbol{H}})\,\boldsymbol{h}_1'(z)=(\boldsymbol{H}-\bar{\boldsymbol{G}})\,\boldsymbol{g}(z). \tag{12}$$

That is, a particular solution for the nonhomogeneous Hilbert problem stated in (9) is

$$\boldsymbol{h}_1'(z)=[\boldsymbol{I}+(\operatorname{Re}\boldsymbol{B}_2)^{-1}\operatorname{Re}\boldsymbol{B}_1]^{-1}\boldsymbol{g}(z), \tag{13}$$

where $\boldsymbol{I}$ is the identity matrix. When compared with the general solution for the counterpart open-crack problem (Deng 1993*a*), it can be observed that the particular solutions for the two problems are identical. This is in fact not a mere coincidence, since it can be shown that this particular solution represents a crack-tip displacement field that is fully continuous, and produces, at the same time, zero tractions along the entire x-axis, which means that the boundary and continuity conditions for both of the cases are satisfied at the same time.

Now, to obtain a general expression for the homogeneous part of the solution for (9), we need to solve a special eigenvalue problem defined by

$$\boldsymbol{U}\boldsymbol{q}=\lambda\boldsymbol{V}\boldsymbol{q}, \tag{14}$$

which has the following characteristic equation

$$|\boldsymbol{U}-\lambda\boldsymbol{V}|=0. \tag{15}$$

Upon expansion of the determinant in (15), we obtain an algebraic equation of the form

$$(\lambda+1)(a\lambda^2-b\lambda+c)=0, \tag{16}$$

where coefficients a, b and c are given by

$$a=-|\boldsymbol{V}|,\quad b=|\boldsymbol{H}|[-H_{21}V_{12}+H_{22}(V_{11}+V_{33})-H_{23}V_{32}],\quad c=|\boldsymbol{U}|. \tag{17}$$

The three eigenvalues can be written as

$$\lambda=\frac{b+\surd(b^2-4ac)}{2a}=\mathrm{e}^{2\pi\epsilon},\quad \lambda^*=-1,\quad \lambda_3=\frac{b-\surd(b^2-4ac)}{2a}=\mathrm{e}^{-2\pi\epsilon_3}, \tag{18}$$

where exponents ϵ and ϵ_3 are related to a, b and c through the following formulas

$$\left.\begin{aligned}\epsilon&=\frac{1}{2\pi}\ln\frac{1-\beta}{1+\beta},\quad \beta=\frac{2a-(b+\surd(b^2-4ac))}{2a+(b+\surd(b^2-4ac))},\\ \epsilon_3&=\frac{1}{2\pi}\ln\frac{1-\beta_3}{1+\beta_3},\quad \beta_3=\frac{2c-(b+\surd(b^2-4ac))}{2c+(b+\surd(b^2-4ac))}.\end{aligned}\right\} \tag{19}$$

Suppose that the eigenvectors associated with eigenvalues λ, λ^* and λ_3 are, respectively, $\boldsymbol{q}$, $\boldsymbol{q}^*$ and $\boldsymbol{q}_3$. Then by expression $\boldsymbol{h}_1'(z)$ in terms of the eigenvectors, the original homogeneous Hilbert problem can be turned into three simple ones, which will allow us to obtain a general homogeneous solution of the form

$$\boldsymbol{h}_1'(z) = \frac{z^{\mathrm{i}\epsilon}\xi(z)}{2\sqrt{(2\pi z)}}\boldsymbol{q} + \xi^*(z)\,\boldsymbol{q}^* + \frac{z^{-\mathrm{i}\epsilon_3}\xi_3(z)}{2\sqrt{(2\pi z)}}\boldsymbol{q}_3, \tag{20}$$

which is for the upper half-plane, where $\xi(z)$, $\xi^*(z)$ and $\xi_3(z)$ are three arbitrary entire functions that take real values when z is replaced by a real-valued variable. Counterpart expression for $\boldsymbol{h}_2'(z)$ of the lower half-plane can be derived from its relation with $\boldsymbol{h}_1'(z)$, but is not listed here for the sake of brevity.

To gain better understanding of the singularities present in (20), the following simplications have been made for expressions in (17). In general, it can be shown that $a = c = H_{22}|\boldsymbol{H}|^2$ and $b = 2a + 4|\boldsymbol{H}|[\mathrm{Im}\,(H_{21}H_{32} - H_{22}H_{31})]^2$, where Im denotes the imaginary part of what follows. Then, because of the positive definiteness of the hermitian matrix $\boldsymbol{H}$, it can be claimed that the constants in (17) are all real-valued with $b \geqslant 2a \geqslant 0$ and that, from (18) and (19), $\beta = \beta_3 \leqslant 0$ and $\epsilon = \epsilon_3 \geqslant 0$, also real-valued.

It can be seen that the first and third terms in (20) give rise to the singular behaviour of the stress state at the crack tip, with $\xi(0)$ and $\xi_3(0)$ as the stress intensity factors. The singularity is of the inverse-square-root type if and only if $\epsilon = \epsilon_3 = 0$, or equivalently, $a = c = \frac{1}{2}b$ or $\mathrm{Im}\,(H_{21}H_{32} - H_{22}H_{31}) = 0$. (For example, the above conditions are satisfied if $\boldsymbol{H}$ is real, such as in the case when the two dissimilar materials are identical. However, this is a trivial case since then the open-crack solution behaves properly. Accordingly we assume $\boldsymbol{H}$ is not real hereafter.) In all other cases, the singularities will be oscillatory but different than those for an open interface crack, in that the two singular terms are in general not related through a pair of complex conjugates. Numerical results for a number of anisotropic bimaterial systems indicate that the non-dimensional mismatch parameter ϵ has a small magnitude, usually on the order of 0.01. For example, for a cubic bicrystal system, let the first material principal direction of each of the two component crystals coincide with the x axis and the other two principal directions with the y and x_3 axes, and then let both crystals rotate an angle of, say 30°, about the x-axis. The resulting bicrystal now behaves like an anisotropic bimaterial in the present coordinate system. For the coupled plane strain/anti-plane shear case, ϵ is found to be 0.0034 for an aluminium/diamond system and 0.027 for a copper/diamond system; and for the coupled plane stress/anti-plane shear case, ϵ is found to be 0.00027 for the aluminium/diamond system and 0.0038 for the copper/diamond system. If the above angle of rotation is different from 30°, or if rotations about other two axes are introduced, the ϵ-value will fluctuate accordingly.

In the next section, we will confine our discussion to a particular class of anisotropic bimaterials, namely those with one plane of symmetry that is aligned such that the in-plane and out-of-plane deformations can be decoupled.

4. Bimaterials with one aligned plane of symmetry

When the bimaterial has a plane of symmetry that is parallel to the xy plane, the in-plane and out-of-plane deformations can be separated from each other in the crack-tip coordinate system, which applies to a broad class of engineering materials, such as aligned orthotropic and transversely isotropic solids, hence is worth a thorough study. In this case, the components of the fourth-order elastic modulus tensor, C_{ijkl}, will be zero when the indices $ijkl$ contain odd numbers of 3. Consequently, it can be shown that matrices $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{G}$, $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ will have zero $3i$ and $i3$ $(i = 1, 2)$ components, so that $\epsilon = \epsilon_3 = 0$. Also, the eigenvectors can be written as

$$\boldsymbol{q} = \left(1, -\frac{H_{21}}{H_{22}}, 0\right), \quad \boldsymbol{q}_3 = (0,0,1), \quad \boldsymbol{q}^* = \left(\frac{\mathrm{i}H_{22}\,\mathrm{Im}\,H_{21}}{H_{11}H_{22}-H_{21}\,\mathrm{Re}\,H_{21}}\cdot 1, 0\right) \tag{21}$$

Note further that the following holds

$$\boldsymbol{Hq} = \bar{\boldsymbol{H}}\bar{\boldsymbol{q}}, \quad \boldsymbol{Hq}_3 = \bar{\boldsymbol{H}}\bar{\boldsymbol{q}}_3, \quad (H_{11}H_{22}-H_{21}\,\mathrm{Re}\,H_{21})\,\boldsymbol{Hq}^* = (H_{11}H_{22}-H_{12}\,\mathrm{Re}\,H_{12})\,\boldsymbol{H}\bar{\boldsymbol{q}}^*. \tag{22}$$

In this case, the solution for the Hilbert problem can be written in a simple form. Omitting the intermediate steps, we list below the complete representation of the crack-tip fields, in terms of $\boldsymbol{h}_1'(z)$ and $\boldsymbol{h}_2'(z)$:

$$\boldsymbol{h}_1'(z) = \frac{\xi(z)}{2\sqrt{(2\pi z)}}\boldsymbol{q} + \frac{(H_{11}H_{22}-H_{21}\,\mathrm{Re}\,H_{21})\,\xi^*(z)}{H_{11}H_{22}}\boldsymbol{q}^* + \frac{\xi_3(z)}{2\sqrt{(2\pi z)}}\boldsymbol{q}_3 + [\boldsymbol{I} + (\mathrm{Re}\,\boldsymbol{B}_2)^{-1}\,\mathrm{Re}\,\boldsymbol{B}_1]^{-1}\boldsymbol{g}(z), \tag{23}$$

$$\boldsymbol{h}_2'(z) = \frac{\xi(z)}{2\sqrt{(2\pi z)}}\bar{\boldsymbol{q}} + \frac{(H_{11}H_{22}-H_{12}\,\mathrm{Re}\,H_{12})\,\xi^*(z)}{H_{11}H_{22}}\bar{\boldsymbol{q}}^* + \frac{\xi_3(z)}{2\sqrt{(2\pi z)}}\boldsymbol{q}_3 + [\boldsymbol{I} + (\mathrm{Re}\,\boldsymbol{B}_1)^{-1}\,\mathrm{Re}\,\boldsymbol{B}_2]^{-1}\boldsymbol{g}(z). \tag{24}$$

We note that (24) can be obtained directly from (23) by replacing H_{21} with H_{12}, and $\boldsymbol{q}$ and $\boldsymbol{q}^*$ with their complex conjugates, and by switching $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$. Since the in-plane and out-of-plane deformations can be uncoupled, we can decompose the last term in each of (23) and (24) into two parts, one of which is for the out-of-plane deformation. For example, for (23) we can get a component of the form $[(H_{33}-G_{33})\,\zeta_3(z)/(2H_{33})]\,\boldsymbol{q}_3$, and correspondingly, a component of the form $[(H_{33}+G_{33})\,\zeta_3(z)/(2H_{33})]\,\boldsymbol{q}_3$ for (24), where $\zeta_3(z)$ is an entire function that becomes imaginary when z is replaced by a real number. These two components, along with the third terms of (23) and (24), represent completely the out-of-plane deformation, and is in fact identical to the counterpart of the open-crack solution (see Deng 1993*a*). The general expressions for $\boldsymbol{h}_1(z)$ and $\boldsymbol{h}_2(z)$ can be obtained from (23) and (24) through simple integrations, and those for $\boldsymbol{f}_1(z)$ and $\boldsymbol{f}_2(z)$ from (2) via a simple inversion. It must be pointed out that since $\xi(z)$, $\xi^*(z)$, $\xi_3(z)$ and $\zeta_3(z)$ are entire functions, one can expand them into Taylor series at the crack tip $z = 0$. By doing so, a Williams-type series expansion of the crack-tip stress and displacement fields will be generated.

The singular behaviour of the crack-tip fields can be clearly seen from (23) and (24). In terms of the real-valued stress intensity factors K and K_3, the singular fields are

$$\boldsymbol{h}_1'(z) = \frac{K}{2\sqrt{(2\pi z)}}\boldsymbol{q} + \frac{K_3}{2\sqrt{(2\pi z)}}\boldsymbol{q}_3, \quad \boldsymbol{h}_2'(z) = \frac{K}{2\sqrt{(2\pi z)}}\bar{\boldsymbol{q}} + \frac{K_3}{2\sqrt{(2\pi z)}}\boldsymbol{q}_3, \tag{25}$$

from which stresses and displacements can be derived through relations in (2) and (1). In general, the similarity between the first term of each of the equations in (25) and that for a classical mode II problem can be clearly seen when H_{21} is zero or an imaginary number. To see this, for example, we observe that the traction along the bimaterial interface, $\boldsymbol{T}(r)$, with components σ_{yx}, σ_{yy} and σ_{y3}, has the following form

$$\boldsymbol{T}(r) \equiv \boldsymbol{t}(r, \theta = 0) = \frac{1}{\sqrt{(2\pi r)}}\left(K, -\frac{\operatorname{Re} H_{21}}{H_{22}} K, K_3\right), \tag{26}$$

where (r, θ) are the polar coordinates associated with (x, y). It is not difficult to see that $\boldsymbol{T}(r)$ is the same as that for a combined, classical mode II and mode III problem if $\operatorname{Re} H_{21}$ is zero, or if H_{21} is zero or an imaginary number, the latter of which is realized when the material is orthotropic. The conclusion is that the crack-tip fields will clearly resemble those for a combined classical mode II and III problem if the bimaterial is orthotropic.

To calculate the elastic energy release rate, G, we note that the crack-tip *sliding* displacement vector, $\delta(r)$, between the two crack surfaces is given by

$$\delta(r) \equiv \boldsymbol{u}(r, \theta = \pi) - \boldsymbol{u}(r, \theta = -\pi) = \sqrt{\left(\frac{2r}{\pi}\right)}\left(\frac{H_{11}H_{22} - H_{12}H_{21}}{H_{22}} K, 0, H_{33}K_3\right). \tag{27}$$

Then with the conventional virtual crack extension method we obtain the following expression:

$$G = \frac{H_{11}H_{22} - H_{12}H_{21}}{4H_{22}} K^2 + \tfrac{1}{4} H_{33} K_3^2, \tag{28}$$

where the coefficients in front of K^2 and K_3^2 are positive since matrix $\boldsymbol{H}$ is positive definite. For stationary and steadily propagating cracks, G can be linked to a path-independent integral. Hence the formula for G given in (28) can be equated to one that is based on the counterpart, open-crack field quantities, for example, the complex stress intensity factor in a small-scale contact situation (see, for example, Suo 1990; Yang *et al.* 1991). However, when the interface crack is propagating unsteadily and conditions for small-scale contact do not exist, (28) is the only appropriate formula for calculating G.

When the bimaterial is orthotropic, the hermitian matrix $\boldsymbol{H}$ has a simple form and its component H_{12} is a pure imaginary number (Suo 1990; Yang *et al.* 1991). In this case, equation (28), in terms of real-valued K and K_3, becomes

$$G = \frac{H_{11}H_{22} - |H_{12}|^2}{4H_{22}} K^2 + \tfrac{1}{4} H_{33} K_3^2. \tag{29}$$

On the other hand, the formula for G from the open-crack singular fields, with K changed to a complex-valued number, is (Yang *et al.* 1991):

$$G = \frac{H_{11}H_{22} - |H_{12}|^2}{4H_{11}} |K|^2 + \tfrac{1}{4} H_{33} K_3^2. \tag{30}$$

We observe that (29) and (30) are almost identical. As such, it can be readily demonstrated that when the crack propagation speed v tends to c_{R}, the coefficient for K^2 in (29) will tend to a finite value, as does the coefficient for $|K|^2$ in (30), which is shown by Yang *et al.* (1991). It is reminded that in the case of a homogeneous material, this coefficient is infinite when $v = c_{\mathrm{R}}$.

5. Summary and closing comments

In this study, we investigated the Comninou-type stress and deformation fields around a stationary or steadily moving interfacial crack, with frictionless crack-surface contact, in an anisotropic bimaterial. Combined in-plane and out-of-plane deformation is considered and a general representation of the crack-tip fields is obtained in terms of certain arbitrary entire functions, which can be expanded into Taylor series to generate a Williams-type series expansion for the crack-tip stresses and displacements. The main points of the results are the following.

(*a*) When the bimaterial has the xy-plane as a plane of symmetry, the combined deformation will have a non-oscillatory singularity of the inverse-square-root type, with the out-of-plane component identical to that for an open interface crack. When the bimaterial is orthotropic, the singularity of the in-plane deformation resembles that for a classical mode II problem, and the formula for the energy release rate differs only slightly from that for an open interface crack, with the coefficient for K^2 tending to a finite limit as v approaches c_{R}.

(*b*) When the in-plane and out-of-plane deformations are not separable, the crack-tip fields may possess oscillatory singularity of a type different than that for a counterpart open-crack problem.

(*c*) In general, the non-singular higher-order terms of the crack-tip fields are composed of terms that are the direct result of the frictionless crack-surface contact condition, and terms that originate from the particular solution of the non-homogeneous Hilbert problem, which generates fully continuous displacement components and zero tractions along the crack surfaces as well as the material interface, and are identical in form with those for the counterpart open-crack problem.

Before closing, it is reminded that the crack tip fields presented in this paper are not applicable to cases where the requirement that $\sigma_{yy} \leqslant 0$ at $\theta = \pm\pi$ is violated. This compression condition at the crack faces is not guaranteed by this asymptotic analysis and must be checked for a particular problem after a solution has been obtained based on the current theory. In this connection, it is noted that the normal traction at $\theta = \pm\pi$ can be expressed, in terms of the singular stress term only, as

$$\sigma_{yy} = -(\operatorname{Im} H_{21}) K / H_{22} \surd(2\pi r). \tag{31}$$

Hence for $\sigma_{yy} \leqslant 0$ to hold along the crack flank one must require that $(\operatorname{Im} H_{21}) K \geqslant 0$.

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