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## APPROXIMATION BY RATIONAL FUNCTIONS

RONALD A. DEVORE

**ABSTRACT.** Making use of the Hardy-Littlewood maximal function, we give a new proof of the following theorem of Pekarski: If  $f'$  is in  $L \log L$  on a finite interval, then  $f$  can be approximated in the uniform norm by rational functions of degree  $n$  to an error  $O(1/n)$  on that interval.

It is well known that approximation by rational functions of degree  $n$  can produce a dramatically smaller error than that for polynomials of degree  $n$ . The best example of this is Newman's theorem [3] which shows that the function  $f(x) = |x|$  can be approximated on  $[-1, 1]$  by rational functions of degree  $n$  to an error  $O(\exp(-c\sqrt{n}))$ , whereas for polynomials of degree  $n$  the error is known to be larger than  $c/n$ . Other authors have shown that such improvement also occurs for certain classes of functions. For example, V. Popov [5] showed that if  $f' \in L_p[0, 1]$ , with  $p > 1$ , then  $r_n(f) = O(n^{-1})$  where  $r_n(f)$  is the error in approximating  $f$  by rational functions  $R$  of degree at most  $n$  in the *uniform norm*:

$$r_n(f) := \inf_{\deg(R)=n} \|f - R\|_\infty[0, 1].$$

To obtain this order of approximation for polynomials requires roughly speaking that  $f' \in L_\infty$ . A striking limiting version of Popov's result was given by A. A. Pekarski [4], who showed that the same conclusion holds when  $f' \in L \log L$ , i.e. if  $|f'| \log(1 + |f'|)$  is integrable.

The Popov and Pekarski proofs of these theorems are quite technical, and it was the purpose of [2] to introduce an elementary technique using maximal functions and partitions of unity for rational functions in order to give a simpler proof of Popov's results. The point of this note is to show that a modification of the technique in [2], albeit a little tricky, will also prove Pekarski's theorem.

The idea in [2] is to partition  $[0, 1]$  into a set  $I$  of disjoint intervals  $I$  and construct associated rational functions  $\psi_I$  which form a partition of unity:  $\sum_{I \in I} \psi_I \equiv 1$ . Our rational approximation  $R$  is then given by

$$(1) \quad R(x) := \sum_{I \in I} f(x_I) \psi_I(x)$$

with  $x_I$  the center of  $I$ . Of course, the intervals  $I$  depend on  $f$ .

The rational functions  $\psi_I$  are constructed using a standard method for partitions of unity. Namely,  $\psi_I := \phi_I / \Phi$  with  $\Phi := \sum \phi_I$ . In the case of Popov's theorem, the  $\phi_I$  depend only on the interval  $I$  and all can be taken of degree 4. The intervals

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$I$  are determined by using the Hardy-Littlewood maximal functions  $M$  which is defined for  $g \in L_1$  by

$$Mg(x) := \sup_{J \ni x} \frac{1}{|J|} \int_J |g|,$$

where the sup is taken over all intervals  $J \subset [0, 1]$  which contain  $x$ .

To prove the Pekarski theorem, we will need to let the degree of  $\phi_I$  depend on  $f$ . The desired properties of  $\phi_I$  are given in the following lemma.

LEMMA 1. For each even integer  $m \geq 8$ , and each interval  $I$  there is a nonnegative rational function  $\phi_I$  of degree at most  $6m$  with the following properties:

- (i)  $\phi_I(x) \geq 1, x \in I,$
- (ii)  $\phi_I(x) \leq 8 \cdot 2^{-\sqrt{\lambda}/4},$  if  $2^{-m/\lambda}|I| \leq \text{dist}(x, I) \leq 1/2$  and  $0 < \lambda < m,$
- (iii)  $\phi_I(x) \leq 4(a^2 + 1)^{-m},$  if  $\text{dist}(x, I) \geq a|I|$  and  $a > 0.$

We postpone the proof of this lemma until the end of the paper. We now use this result to prove the following.

THEOREM. There is an absolute constant  $c > 0$  such that for  $n = 1, 2, \dots$

$$r_n(f) \leq c \|M(f')\|_1 n^{-1}, \quad n = 1, 2, \dots,$$

whenever  $M(f')$  is in  $L_1[0, 1].$

REMARK: It is well known (see e.g. [1]) that  $g \in L \log L$  is equivalent to  $M(g) \in L_1$  and therefore this theorem is equivalent to Pekarski's.

PROOF. It is enough to consider functions  $f$  with  $\|M(f')\|_1 = 1.$  It follows that  $\|f'\|_1 \leq 1$  and hence there is a collection  $I$  of at most  $n$  intervals  $I$  which are a disjoint partition of  $[0, 1]$  and satisfy

$$(2) \quad \frac{1}{n} \leq \int_I |f'| \leq \frac{2}{n}, \quad I \in I.$$

For each  $I \in I,$  we let  $m_I$  be the smallest integer which is both larger than 7 and also larger than  $4n \int_I M(f').$  If  $\phi_I$  is the function of Lemma 1 for the interval  $I$  and for  $m = m_I,$  we let  $\Phi := \sum_{I \in I} \phi_I.$  By Lemma 1,  $\Phi \geq 1,$  on  $[0, 1]$  and hence the functions  $\psi_I$  satisfy

$$(3) \quad \psi_I(x) \leq \phi_I(x), \quad 0 \leq x \leq 1.$$

We now take  $R$  as in (1) with  $x_I$  the center of  $I.$  Since  $\sum m_I \leq 16n,$   $R$  has degree  $\leq 96n.$  To estimate  $|f(x) - R(x)|,$  we let  $I_0$  denote the interval of  $I$  which contains  $x; I_1$  the interval of  $I$  immediately to the right of  $I_0; I_{-1}$  the interval immediately to the left of  $I_0;$  and so on. We have

$$(4) \quad f(x) - R(x) = \sum_{I \in I} (f(x) - f(x_I)) \psi_I(x) =: \sum_{-1} + \sum_0 + \sum_1$$

Where  $\sum_{-1}$  denotes the sum over those  $I = I_k$  with  $k < -1,$   $\sum_1$  the sum over those  $I = I_k$  with  $k > 1$  and  $\sum_0$  the sum of the terms  $k = -1, 0, 1.$  Clearly,  $|f(x) - f(x_{I_k})| \leq 2(|k| + 1)/n.$  Since the  $\psi_I$  are nonnegative and add up to one, we have

$$(5) \quad \sum_0 \leq 12/n.$$

The estimates for  $\sum_{-1}$  and  $\sum_1$  are the same and therefore we estimate only  $\sum_1$ . For this, we fix  $k > 1$  and estimate the term in  $\sum_1$  corresponding to  $I = I_k$ . We have

$$(6) \quad e_k := |f(x) - f(x_I)|\psi_I(x) \leq \frac{2(k+1)}{n}\psi_I(x) \leq \frac{4k}{n}\phi_I(x).$$

We write  $\text{dist}(x, I) =: a|I|$ , with  $a \geq 0$ , and we consider three cases.

*Case  $a \geq \sqrt{k}$ .* Then since  $m \geq 8$ , by (iii) of Lemma 1, we have  $\psi_I(x) \leq \phi_I(x) \leq 4k^{-4}$  and consequently

$$(7) \quad e_k \leq 16k^{-3}n^{-1}.$$

*Case  $1/2 \leq a \leq \sqrt{k}$ .* The smallest interval  $J$  which contains  $x$  and  $I$  has length  $(a+1)|I|$  and on  $I$ ,

$$M(f') \geq \frac{1}{|J|} \int_J |f'| \geq \frac{k}{n(a+1)|I|}$$

and therefore  $m \geq 4n \int_I M(f') \geq 4k/(a+1) \geq \sqrt{k}$ . This gives by (iii) of Lemma 1,

$$(8) \quad e_k \leq \frac{4k}{n}\phi_I(x) \leq \frac{4k}{n}(a^2+1)^{-m} \leq \frac{16k}{n}(5/4)^{-\sqrt{k}}.$$

*Case  $0 < a < 1/2$ .* We write  $a =: 2^{-m/\lambda}$  with  $0 < \lambda < m$ . Similar to the second case, for  $u \in I$ , we have  $M(f')(u) \geq (k-1)/n(u-x)$ . Therefore,

$$m \geq 4n \int_I M(f') \geq 4(k-1) \int_{2^{-m/\lambda}|I|}^{|I|} \frac{du}{u} \geq 2k \left(\frac{m}{\lambda}\right) \log 2.$$

This shows that  $\lambda \geq 2k \log 2 \geq k$ . Hence by (ii) of Lemma 1, we have

$$(9) \quad e_k \leq \frac{4k}{n}\phi_I(x) \leq \frac{32k}{n}2^{-\sqrt{k}/4}.$$

The estimates (7)–(9) serve to show that  $\sum_1 = \sum e_k \leq cn^{-1}$ , with  $c$  an absolute constant. This combined with (5) and the corresponding estimate for  $\sum_{-1}$  when placed in (4) proves the theorem.

We turn now to the proof of Lemma 1. For this, we shall use the following:

**LEMMA 2.** *For each even integer  $m \geq 8$  there is a rational function  $R$  of degree  $\leq 2m$  with the following properties:*

- (i)  $R(x) \geq 1, x \in [-1, 0]$ ,
- (ii)  $0 \leq R(x) \leq 2$ , for  $-\infty < x < \infty$ ,
- (iii)  $|R(x)| \leq 2 \cdot 2^{-m/4j}$ , if  $2^{-(j+1)^2/m} \leq x \leq 1/2$ , with  $\sqrt{m} - 1 \leq j < m$ .

**PROOF.** With  $a := 2^{-1/m}$  and  $a_k := a^{k^2}$ , we define  $p(x) := \prod_1^m (x + a_k)$ . We first estimate  $\pi(x) := p(-x)/p(x) = \prod_1^m (-x + a_k)/(x + a_k)$  when  $x \geq 0$ . Since each term in  $\pi$  has absolute value at most 1, we have

$$(10) \quad |\pi(x)| \leq 1, \quad x \geq 0.$$

When  $a_m \leq x \leq 1/2$ , we take  $j$  so that  $a_{j+1} \leq x \leq a_j$ ; so  $\sqrt{m} - 1 \leq j < m$ . Then,

$$|\pi(x)| \leq \pi_1(x) := \prod_1^j \frac{a_k - x}{a_k + x}.$$

We now use the inequality  $(1 - t)/(1 + t) \leq e^{-2t}$ , which is valid for  $0 \leq t \leq 1$ . This gives

$$(11) \quad |\pi(x)| \leq \prod_1^j \frac{1 - x/a_k}{1 + x/a_k} \leq \exp\left(-2 \sum_1^j \frac{a_{j+1}}{a_k}\right) =: \exp(-2\sigma(j)).$$

Since  $(j + 1)^2 - k^2 \leq (j - k + 1)(2j + 1)$ , we have with  $b := a^{2j+1}$ ,

$$\sigma(j) \geq \sum_1^j b^\nu = b \frac{1 - b^j}{1 - b}.$$

But, since  $\sqrt{m} - 1 \leq j < m$ ,  $b \geq 1/4$ ;  $1 - b^j \geq 1/2$ ; also  $1 - e^{-t} \leq t$ , for  $0 < t \leq 1$ . Hence,

$$\sigma(j) \geq \frac{m}{8(2j + 1) \log 2}.$$

Since  $2 \log 2 \leq 1/\log 2$ , using our last estimate for  $\sigma(j)$  in (11) gives

$$(12) \quad |\pi(x)| \leq \exp\left(\frac{-m \log 2}{2(2j + 1)}\right) \leq 2^{-m/8j}, \quad a_{j+1} \leq x \leq a_j, \text{ for } \sqrt{m} - 1 \leq j < m.$$

We can now take  $R(x) := 2\pi^2(x)/(1 + \pi^2(x))$ . Since  $\pi(-x) = 1/\pi(x)$  and  $R(-x) = 2/(1 + \pi^2(x))$ , (i) follows from (10). The estimate (ii) is obvious, while (iii) follows immediately from (12).

PROOF OF LEMMA 1. It is enough to consider  $I = [-1, 0]$  since the lemma then follows for any other interval by a change of scale. We let

$$T(x) := ((x + 1/2)^2 + 3/4)^{-m}$$

and  $R$  be as in Lemma 2. We can then take  $\phi(x) := R(x)R(-x)T(x)$ . Since  $T(x) \geq 1$ ,  $x \in I$ , (i) follows from (i) of Lemma 2. Since  $T(x) \leq 1$ ,  $x \notin I$ , (ii) follows when  $\lambda \leq 4$  from Lemma 2(ii). For the other values of  $\lambda$ , we choose  $j$  so that  $j^2 < m^2/\lambda < (j + 1)^2$ , and then (ii) follows from Lemma 2(ii), (iii). Finally, if  $\text{dist}(x, I) \geq a$ , then  $(x - 1/2)^2 + 3/4 \geq (a + 1/2)^2 + 3/4 \geq a^2 + 1$  and therefore (iii) follows from (ii) of Lemma 2.

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