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Intersecting Chains in Finite Vector Spaces

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We prove an Erdős–Ko–Rado-type theorem for intersecting k-chains of subspaces of a finite vector space. This is the q-generalization of earlier results of Erdős, Seress and Székely for intersecting k-chains of subsets of an underlying set. The proof hinges on the author's proper generalization of the shift technique from extremal set theory to finite vector spaces, which uses a linear map to define the generalized shift operation. The theorem is the following.

For $c = 0, 1$, consider k-chains of subspaces of an n-dimensional vector space over $GF(q)$, such that the smallest subspace in any chain has dimension at least c, and the largest subspace in any chain has dimension at most $n - c$. The largest number of such k-chains under the condition that any two share at least one subspace as an element of the chain, is achieved by the following constructions:

(1) fix a subspace of dimension c and take all k-chains containing it,
(2) fix a subspace of dimension $n - c$ and take all k-chains containing it.

1. Introduction

P. L. Erdős, Á. Seress and L. Székely recently proved [3] an Erdős–Ko–Rado-type theorem for families of intersecting k-chains in a Boolean algebra and subsequently generalized it to families of intersecting k-chains in truncated Boolean algebras [4]. Their proofs used shifting at a crucial step. Although P. L. Erdős, Á. Seress and L. Székely contemplated generalization of their results to families of intersecting k-chains of subspaces of a vector space over a finite field, without an appropriate shifting in vector spaces the generalization has eluded them.

In this paper we present the generalization of the results of Erdős, Seress and Székely to chains of subspaces. Although we use many of their ideas, specializing the proof presented

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here to sets, one obtains a proof for the set case which is somewhat different from theirs and can be considered as a new proof.

The main result of the paper is the following theorem.

For $c = 0, 1$, consider k-chains of subspaces of an n-dimensional vector space over $GF(q)$, such that the smallest subspace in any chain has dimension at least c, and the largest subspace in any chain has dimension at most $n - c$. The largest number of such k-chains under the condition that any two share at least one subspace as an element of the chain, is achieved by the following constructions:

(1) *fix a subspace of dimension c and take all k-chains containing it,*
(2) *fix a subspace of dimension $n - c$ and take all k-chains containing it.*

In Section 2 of this paper we give the notations and definitions used. For the sake of completeness we define the chain shift on sets in Section 3, as it was defined in [3]. Section 4 contains the definition of the chain shift on linear spaces. In Section 5 we show the existence of intersecting families of k-chains, which have maximum cardinality and are invariant under the shift operation. We call these invariant structures saturated. We also show that a saturated intersecting family of k-chains of subspaces is standard intersecting, that is, any two chains intersect in an element of the set $\{V_0, V_1, V_2 \ldots, V_n\}$, where V_i is a prefixed i-dimensional subspace of our n-dimensional space. Section 6 contains a few technical lemmas. In Section 7 we describe a natural correspondence between maximal families of k-chains of subsets and subspaces which are invariant under the corresponding shift operation, and we also show that maximal saturated families and maximal standard intersecting families are the same. Section 8 contains a few observations about standard intersecting families, and in Section 9 we prove our main result.

2. Notations and definitions

Let $GF(q)$ be the finite field with q elements, and $V = V_n$ be an n-dimensional vector space over $GF(q)$. For a set of vectors H the linear span of H is denoted by $\langle H \rangle$. A k-*subspace* is a subspace with dimension k, a k-*set* is a k-element set, a k-*chain* is a linearly ordered k-element set (in this paper, the ordering will be by inclusion). A family of chains is *intersecting* if every two chains from the family have a common element.

For a nonnegative integer i, the set $\{1, 2, \ldots, i\}$ will be denoted by $[i]$. Note that $[0] = \emptyset$. An *initial segment* is any set of the form $[i]$.

For convenience we use the following notation throughout this paper. Numbers will be denoted by lower-case letters, vectors will be denoted by overlined lower-case letters, and scalars from $GF(q)$ will be denoted by lower-case Greek letters. Subspaces of V will be denoted by upper-case letters and subsets of $[n]$ will be denoted by underlined lower-case letters. Chains of subspaces and chains of subsets of $[n]$ will be denoted by upper-case Greek and underlined lower-case Greek letters, respectively. Families of chains of subspaces and subsets of $[n]$ will be denoted by upper-case and lower-case Gothic letters, respectively. In general, in most of the cases structures concerning subsets of

$[n]$ are denoted by underlined lower-case letters of some style, and the corresponding structures of vectors/subspaces will be denoted by the corresponding upper-case letters.

The set of k-chains of subspaces of V is denoted by $\mathfrak{B}_{n,k}$, and the set of k-chains of subsets of $[n]$ is denoted by $\mathfrak{b}_{n,k}$. For an integer $c \geqslant 0$ we denote by $\mathfrak{B}^c_{n,k}$ and $\mathfrak{b}^c_{n,k}$ the families

$$\mathfrak{B}^c_{n,k} = \{\Gamma = (L^1 \subset L^2 \subset \cdots \subset L^k) \; : \; \Gamma \in \mathfrak{B}_{n,k}, \dim(L^1) \geqslant c, \dim(L^k) \leqslant n - c\}$$

and

$$\mathfrak{b}^c_{n,k} = \{\underline{\gamma} = (\underline{l}^1 \subset \underline{l}^2 \subset \cdots \subset \underline{l}^k) \; : \; \underline{\gamma} \in \mathfrak{b}_{n,k}, \left|\underline{l}^1\right| \geqslant c, \left|\underline{l}^k\right| \leqslant n - c\},$$

respectively. Note that $\mathfrak{B}^0_{n,k} = \mathfrak{B}_{n,k}$ and $\mathfrak{b}^0_{n,k} = \mathfrak{b}_{n,k}$; note also that, if $\mathfrak{B}^c_{n,k}$ and $\mathfrak{b}^c_{n,k}$ are nonempty, then $k \leqslant n - 2c + 1$.

For integers n, c such that $n \geqslant 2c$ let us define the operation $\widehat{\ }$ as follows: for every set $\underline{m} \subseteq [n]$,

$$\widehat{\underline{m}} = \underline{m} \setminus \{n - c\}.$$

For notational convenience we denote $\widehat{[i]}$ by $\left[\,\widehat{i}\,\right]$. $\mathfrak{b}^c_{\widehat{n},k}$ is defined analogously to $\mathfrak{b}^c_{n,k}$ on the underlying set $[\widehat{n}]$. Obviously $[\widehat{n-c}] = \left[\,\widehat{n-c-1}\,\right]$, $\widehat{[n]}$ and $[n-1]$ have $n-1$ elements and their power sets are isomorphic. Moreover $\widehat{\underline{m}} \subseteq \widehat{[n]}$ and $\left|\underline{m}\right| - 1 \leqslant \left|\widehat{\underline{m}}\right| \leqslant \left|\underline{m}\right|$. For a k-chain $\underline{\gamma} \in \mathfrak{b}^c_{n,k}$, such that $\underline{\gamma} = (\underline{m}^1 \subset \underline{m}^2 \subset \cdots \subset \underline{m}^k)$, let $i, 1 \leqslant i \leqslant k+1$ be the unique integer for which we have $n - c \notin \underline{m}^p$ if and only if $p < i$. After this preparation we define $\widehat{\underline{\gamma}}$ as follows:

$$\widehat{\underline{\gamma}} = \begin{cases} (\widehat{\underline{m}^1} \subset \cdots \subset \widehat{\underline{m}^{i-1}} \subset \widehat{\underline{m}^{i+1}} \cdots \subset \widehat{\underline{m}^k}), & \text{if } i \leqslant k \text{ and } \underline{m}^i = \underline{m}^{i-1} \cup \{n-c\}, \\ (\widehat{\underline{m}^1} \subset \widehat{\underline{m}^2} \subset \cdots \subset \widehat{\underline{m}^k}), & \text{otherwise.} \end{cases}$$

Obviously, if $n - c \notin \underline{m}^1$ and $\left|\underline{m}^k \setminus \{n-c\}\right| \leqslant n - c - 1$, then $\widehat{\underline{\gamma}} \in \mathfrak{b}^c_{\widehat{n},k} \cup \mathfrak{b}^c_{\widehat{n},k-1}$.

For an $\mathfrak{r} \subseteq \mathfrak{b}^c_{n,k}$ we define the set $\widehat{\mathfrak{r}}$ as follows:

$$\widehat{\mathfrak{r}} = \{\widehat{\underline{\gamma}} \; : \; \underline{\gamma} \in \mathfrak{r}\}.$$

Note that $\widehat{\mathfrak{r}}$ can contain both k- and $(k-1)$-chains, and these chains might not all be from $\mathfrak{b}^c_{\widehat{n},k} \cup \mathfrak{b}^c_{\widehat{n},k-1}$.

For a nonnegative integer l, for a given $\underline{\gamma} \in \mathfrak{b}_{n,l}$, where $\underline{\gamma} = (\underline{m}^1 \subset \underline{m}^2 \subset \cdots \subset \underline{m}^l)$, the *gap* of $\underline{\gamma}$ is the number

$$\mathfrak{g}(\underline{\gamma}) = \sum_{p=1}^{l} \sum_{r \in \underline{m}^p \setminus \underline{m}^{p-1}} \left|[r] \setminus \underline{m}^p\right|, \quad \text{where } \underline{m}^0 = \emptyset.$$

Note that the gap of the empty chain is 0. The gap is obviously nonnegative, and it is zero if and only if the chain is empty or it consists entirely of initial segments.

For a set $\underline{m} \subseteq \left[\,n\right]$, assume that $\underline{m} = \{i_1 < i_2 < \cdots < i_t\}$. The *gap* of $\underline{m}$ will be defined as the gap of the 1-chain containing $\underline{m}$, that is,

$$\mathfrak{g}(\underline{m}) = \mathfrak{g}\big((\underline{m})\big) = \sum_{s=1}^{t}(i_s - s) = \left(\sum_{i \in \underline{m}} i\right) - \frac{t(t+1)}{2}.$$

Note that, for a chain $\underline{\gamma} = (\underline{l}^1 \subset \cdots \subset \underline{l}^k)$, we have that in general $\mathfrak{g}(\underline{\gamma}) \neq \mathfrak{g}(\underline{l}^1) + \mathfrak{g}(\underline{l}^2 \setminus \underline{l}^1) + \cdots + \mathfrak{g}(\underline{l}^k \setminus \underline{l}^{k-1})$.

We also introduce the following notations: for nonnegative integers n, k, c, i, where $1 \leqslant k \leqslant n+1-2c$ and $c \leqslant i \leqslant n-c$, we define the family of k-chains of subsets $\mathfrak{t}^c_{n,k}[i]$ by

$$\mathfrak{t}^c_{n,k}[i] = \{\underline{\gamma} \in \mathfrak{b}^c_{n,k} \ : \ [i] \in \underline{\gamma}\}.$$

We will also use the notation $\mathfrak{t}_{n,k}[i]$ for the family $\mathfrak{t}^0_{n,k}[i]$. Obviously $\mathfrak{t}^c_{n,k}[i] \subseteq \mathfrak{b}^c_{n,k}$, and it is an intersecting family of k-chains of subsets.

For the rest of this paper $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n$ will be a fixed increasing sequence of subspaces of V such that $\dim(V_i) = i$. An *initial space* will be any of the V_is. For a positive integer j such that $j \leqslant n$ we denote the set $V_j \setminus V_{j-1}$ by $\mathscr{W}_j$. The *index* of a nonzero vector $\overline{v}$ is denoted by $I(\overline{v})$, and $I(\overline{v}) = j$ if and only if $\overline{v} \in \mathscr{W}_j$. Obviously $|V_j| = q^j$ for every j, $0 \leqslant j \leqslant n$, and if $j \neq 0$ then $|\mathscr{W}_j| = q^j - q^{j-1} = q^{j-1}(q-1)$.

For nonnegative integers n, k, c, i, where $1 \leqslant k \leqslant n+1-2c$ and $c \leqslant i \leqslant n-c$, we define the set $\mathfrak{T}^c_{n,k}[i]$ as the family

$$\mathfrak{T}^c_{n,k}[i] = \{\Gamma \in \mathfrak{B}^c_{n,k} \ : \ V_i \in \Gamma\}.$$

We will also use the notation $\mathfrak{T}_{n,k}[i]$ for $\mathfrak{T}^0_{n,k}[i]$. Obviously $\mathfrak{T}^c_{n,k}[i] \subseteq \mathfrak{B}^c_{n,k}$, and it is an intersecting family of k-chains of subspaces.

The *profile of a subspace* U, denoted by $\underline{p}(U)$, is the set $\{j : U \cap \mathscr{W}_j \neq \emptyset\}$. The profile of V_0 is obviously $\emptyset$. Clearly, if $U \subset W$ then $\underline{p}(U) \subset \underline{p}(W)$.

The *profile of a k-chain of subspaces* $\Gamma = (L^1 \subset L^2 \subset \cdots \subset L^k)$ is $\underline{\pi}(\Gamma) = (\underline{p}(L^1) \subset \underline{p}(L^2) \subset \cdots \subset \underline{p}(L^k))$. Note that for a k-chain of subspaces Γ we have that if $\Gamma \in \mathfrak{B}^c_{n,k}$, then $\underline{\pi}(\Gamma) \in \mathfrak{b}^c_{n,k}$.

For $\mathfrak{R} \subseteq \mathfrak{B}^c_{n,k}$, we define $\mathfrak{h}_{\mathfrak{R}}$ as $\mathfrak{h}_{\mathfrak{R}} = \{\underline{\pi}(\Gamma) \ : \ \Gamma \in \mathfrak{R}\}$. Obviously $\mathfrak{h}_{\mathfrak{R}} \subseteq \mathfrak{b}^c_{n,k}$, and if $\mathfrak{R}$ is intersecting, then so is $\mathfrak{h}_{\mathfrak{R}}$.

For $\mathfrak{r} \subseteq \mathfrak{b}^c_{n,k}$, then we define $\mathfrak{H}_{\mathfrak{r}}$ as $\mathfrak{H}_{\mathfrak{r}} = \{\Gamma \in \mathfrak{B}^c_{n,k} \ : \ \underline{\pi}(\Gamma) \in \mathfrak{r}\}$. $\mathfrak{H}_{\mathfrak{r}} \subseteq \mathfrak{B}^c_{n,k}$, but the intersecting property of $\mathfrak{r}$ does not ensure in general that $\mathfrak{H}_{\mathfrak{r}}$ will be intersecting.

It is easy to check that $\mathfrak{h}_{\mathfrak{T}^c_{n,k}[i]} = \mathfrak{t}^c_{n,k}[i]$, and $\mathfrak{H}_{\mathfrak{t}^c_{n,k}[i]} = \mathfrak{T}^c_{n,k}[i]$.

A set of nonzero vectors $\{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_s\}$ is an *essential set* if for every i, j such that $1 \leqslant i < j \leqslant s$ we have $I(\overline{v}_i) \neq I(\overline{v}_j)$.

Assume without loss of generality that for $i < j$ we have $I(\overline{v}_i) < I(\overline{v}_j)$. Then, since $\langle \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_l \rangle \subseteq V_{I_{\overline{v}_l}}$, we have that, for $p : l < p \leqslant s$, $v_p \notin \langle \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_l \rangle$. This implies that every essential set is linearly independent.

For a subspace U, we say that the set of vectors $\{\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_m\}$ is an *essential basis* of U if it is an essential set which is a basis of U.

Let U be a subspace of V. If $\mathscr{G}$ is an essential set for U, then let $\mathscr{H}$ be another essential set such that $\underline{p}(\mathscr{H} = \underline{p}(U) \setminus \underline{p}(\mathscr{G})$. It follows the $\mathscr{G} \cup \mathscr{H}$ is an esential (therefore linearly independent) set for U, which is a basis of U. This implies that every essential set for U can be extended to an essential basis of U; moreover, if $\{\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_m\}$ is an essential basis of U, then $\underline{p}(U) = \{I(\overline{g}_1), I(\overline{g}_2), \ldots, I(\overline{g}_k)\}$.

For the rest of the paper, whenever we have an essential basis $\{\overline{g}_1, \ldots, \overline{g}_k\}$, we will always assume that $I(\overline{g}_1) < I(\overline{g}_2) < \cdots < I(\overline{g}_k)$.

If $\mathscr{X} = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$ is an essential basis of V, then a subspace U is $(j, \mathscr{X})$-*free* if

$U \subseteq \langle \mathscr{X} \setminus \{\overline{x}_j\}\rangle$. Similarly, a vector $\overline{v} \in V$ is $(j,\mathscr{X})$-free if $\overline{v} \in \langle \mathscr{X} \setminus \{\overline{x}_j\}\rangle$. Note that the only vector that is both $(j,\mathscr{X})$-free and a multiple of $\overline{x}_j$ is the zero vector.

We define a family of linear transformations $\phi_{j,\mathscr{X}}$ as follows. For an essential basis $\mathscr{X}$ and an integer j, $1 \leqslant j \leqslant n$, let $\phi_{j,\mathscr{X}}$ be the linear extension of the map which assigns the zero vector to $\overline{x}_j$ and leaves the other elements of $\mathscr{X}$ fixed, that is, for a $\overline{v} \in V$ such that $\overline{v} = \sum_i \alpha_i \overline{x}_i$,

$$\phi_{j,\mathscr{X}}\left(\sum_{i=1}^{n} \alpha_i \overline{x}_i\right) = \sum_{i:i\neq j} \alpha_i \overline{x}_i.$$

It is obvious that the kernel of $\phi_{j,\mathscr{X}}$ is $\langle \overline{x}_j\rangle$, and for a subspace U we have $\phi_{j,\mathscr{X}}(U) = U$ precisely when U is $(j,\mathscr{X})$-free. Also $\dim(U) > \dim(\phi_{j,\mathscr{X}}(U))$ is equivalent to $\overline{x}_j \in U$, in which case $\phi_{j,\mathscr{X}}(U) \subset U$. For any vector $\overline{v}$ such that $\overline{v} \notin \langle \overline{x}_j\rangle$ we have that if $I(\overline{v}) \neq j$ then $I(\overline{v}) = I\big(\phi_{j,\mathscr{X}}(\overline{v})\big)$, and if $I(\overline{v}) = j$ then $I(\overline{v}) > I\big(\phi_{j,\mathscr{X}}(\overline{v})\big)$.

If $\Gamma = (L^1 \subset L^2 \subset \cdots \subset L^k) \in \mathfrak{B}^c_{n,k}$, define $(j,\mathscr{X})$-*low* of Γ as

$$\mathrm{Low}_{j,\mathscr{X}}(\Gamma) = \{L^i \in \Gamma \ : \ L^i \text{ is } (j,\mathscr{X})\text{-free}\}.$$

Clearly $\mathrm{Low}_{j,\mathscr{X}}(\Gamma)$ is a subchain of Γ. Since subspaces of a $(j,\mathscr{X})$-free subspace must also be $(j,\mathscr{X})$-free, there is a nonnegative integer m_1 such that $m_1 \leqslant k$ and $L^i \in \Gamma$ is $(j,\mathscr{X})$-free if and only if $i \leqslant m_1$.

Define the $(j,\mathscr{X})$-*high* of Γ as

$$\mathrm{High}_{j,\mathscr{X}}(\Gamma) = \{L^i \in \Gamma \ : \ \overline{x}_j \in L^i\}.$$

Clearly $\mathrm{High}_{j,\mathscr{X}}(\Gamma)$ is a subchain of Γ. Since a subspace that does not contain $\overline{x}_j$ can not contain another subspace that contains $\overline{x}_j$, there is a positive integer m_2 such that $m_2 \leqslant k+1$, and $L^i \in \Gamma$ is $(j,\mathscr{X})$-free if and only if $i \geqslant j$.

Finally, define the $(j,\mathscr{X})$-*middle* of Γ as

$$\mathrm{Mid}_{j,\mathscr{X}}(\Gamma) = \{L^i \in \Gamma \ : \ \overline{x}_j \notin L^i \text{ and } L^i \text{ is not } (j,\mathscr{X})\text{-free}\}.$$

$\mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ is a subchain of Γ, and by our previous observations the elements of $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ contain the elements of $\mathrm{Low}_{j,\mathscr{X}}(\Gamma)$, and are also contained in the elements of $\mathrm{High}_{j,\mathscr{X}}(\Gamma)$ (provided the sets are nonempty).

3. Definition of chain shift on sets

For the sake of completeness, before we define the chain shift on finite vector spaces, we include the definition of the chain shift on sets. Here we follow [3].

Let $\mathfrak{r}$ be a family of pairwise intersecting k-chains from $\mathfrak{b}^c_{n,k}$, and let $1 \leqslant i < j \leqslant n$ be integers. The (i,j)-*shift* $\mathfrak{r}_{i,j}$ of the family $\mathfrak{r}$ is defined as follows.

For every k-chain $\underline{\gamma} = (\underline{m}^1 \subset \cdots \subset \underline{m}^k)$ let $\underline{\gamma}_{i,j} = (\underline{m}^1_{i,j} \subset \cdots \subset \underline{m}^k_{i,j})$, where

$$\underline{m}_{i,j} = \begin{cases} (\underline{m} \setminus \{j\}) \cup \{i\}, & \text{if } j \in \underline{m} \text{ and } i \notin \underline{m}, \\ \underline{m}, & \text{otherwise.} \end{cases}$$

We say that $\underline{m}_{i,j}$ is the (i, j)-shift of $\underline{m}$. Since shifting preserves containment, $\underline{\gamma}_{i,j}$ is a k-chain. Then the (i, j)-shift of $\mathfrak{r}$ is obtained by the following:

$$\mathfrak{r}_{i,j} = \{\underline{\gamma}_{i,j} : \underline{\gamma} \in \mathfrak{r} \text{ and } \underline{\gamma}_{i,j} \notin \mathfrak{r}\} \cup \{\underline{\gamma} : \underline{\gamma} \in \mathfrak{r} \text{ and } \underline{\gamma}_{i,j} \in \mathfrak{r}\}.$$

We call an intersecting family $\mathfrak{r}$ *compressed* if it is invariant under every (i, j)-shift.

It has been proved in [3] that the (i, j)-shift preserves the intersecting property and the cardinality of a family of pairwise intersecting k-chains.

We introduce the following definition. A family of k-chains of subsets is *standard intersecting* if the intersection of two chains from the family contains an initial segment. If the family contains only one chain, then we call it standard intersecting if the chain is nonempty and it contains at least one initial segment.

In [3] it has been proved that a compressed intersecting family is standard intersecting.

The (i, j)-shift $\underline{m}_{i,j}$ of a set $\underline{m}$ can be viewed as follows. We define a function $f_{i,j} : [n] \to [n]$ such that

$$f_{i,j}(l) = \begin{cases} l, & \text{if } l \neq j, \\ i, & \text{if } l = j \end{cases}$$

and it is easy to see that

$$\underline{m}_{i,j} = \begin{cases} f_{i,j}(\underline{m}), & \text{if } |\underline{m}| = |f_{i,j}(\underline{m})|, \\ \underline{m}, & \text{otherwise.} \end{cases}$$

4. Definition of chain shift on vector-spaces

We define the chain shift on vector spaces similarly to the chain shift on sets. $\phi_{j,\mathscr{X}}$ is the linear transformation which takes the place of $f_{i,j}$ in the shift operation, and the dimension of the subspace takes the place of the cardinality of a subset. Clearly, $\dim(U) \geqslant \dim(\phi_{j,\mathscr{X}}(U))$, and equality holds if and only if $\overline{x}_j \notin U$. As before, for any vector space U we define $U_{j,\mathscr{X}}$ as

$$U_{j,\mathscr{X}} = \begin{cases} \phi_{j,\mathscr{X}}(U), & \text{if } \overline{x}_j \notin U, \\ U, & \text{otherwise.} \end{cases}$$

Obviously $\dim(U_{j,\mathscr{X}}) = \dim(U)$, and $U = U_{j,\mathscr{X}}$ precisely when U is either $(j, \mathscr{X})$-free or $\overline{x}_j \in U$. Moreover, if $W \subset U$, then $W_{j,\mathscr{X}} \subset U_{j,\mathscr{X}}$. (If $\overline{x}_j$ is in both of U, W or neither then this is obvious. If $x_j \in U \setminus W$, then $W_{j,\mathscr{X}} = \phi_{j,\mathscr{X}}(W) \subseteq \phi_{j,\mathscr{X}}(U) \subset U = U_{j,\mathscr{X}}$.)

If Ω is a set of subspaces, then we define $\Omega_{j,\mathscr{X}}$ as the set $\{U_{j,\mathscr{X}} : U \in \Omega\}$. From these we have that, whenever $\Gamma = (L^1 \subset L^2 \subset \cdots \subset L^k) \in \mathfrak{B}^c_{n,k}$, then

$$\Gamma_{j,\mathscr{X}} = (L^1_{j,\mathscr{X}} \subset L^2_{j,\mathscr{X}} \subset \cdots \subset L^k_{j,\mathscr{X}})$$

is also an element of $\mathfrak{B}^c_{n,k}$, and $\Gamma_{j,\mathscr{X}} \cap \Gamma = \mathrm{Low}_{j,\mathscr{X}}(\Gamma) \cup \mathrm{High}_{j,\mathscr{X}}(\Gamma)$; but $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma_{j,\mathscr{X}}) = \emptyset$ and $\mathrm{Low}_{j,\mathscr{X}}(\Gamma_{j,\mathscr{X}}) = \mathrm{Low}_{j,\mathscr{X}}(\Gamma) \cup \{L_{j,\mathscr{X}} : L \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)\}$.

If $\mathfrak{R}$ is an intersecting family of k-chains of subspaces, then let

$$\mathfrak{A}_{j,\mathscr{X},\mathfrak{R}} = \{\Gamma : \Gamma \in \mathfrak{R}, \Gamma_{j,\mathscr{X}} \notin \mathfrak{R}, \text{ and for any } \Delta \in \mathfrak{R}, \ \Delta_{j,\mathscr{X}} = \Gamma_{j,\mathscr{X}} \text{ implies } \Delta = \Gamma\}$$

and

$$\mathfrak{D}_{j,\mathscr{X},\mathfrak{R}} = \{\Gamma_{j,\mathscr{X}} : \Gamma \in \mathfrak{R}\}.$$

Since for $\Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}$ we have $\Gamma \neq \Gamma_{j,\mathscr{X}}$, it follows that $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma) \neq \emptyset$. Also from the definition we have that $\{\Gamma_{j,\mathscr{X}} : \Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}\} \cap \mathfrak{R} = \emptyset$ and $\{\Gamma_{j,\mathscr{X}} : \Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}\} \subseteq \mathfrak{D}_{j,\mathscr{X},\mathfrak{R}}$. Moreover, if Γ, Δ are two different k-chains from $\mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}$ then $\Gamma_{j,\mathscr{X}} \neq \Delta_{j,\mathscr{X}}$; therefore $|\mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}| = |\{\Gamma_{j,\mathscr{X}} : \Gamma_{j,\mathscr{X}} \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}\}|$. Let us define

$$\mathfrak{C}_{j,\mathscr{X},\mathfrak{R}} = \left(\mathfrak{R} \setminus \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}\right) \cup \mathfrak{D}_{j,\mathscr{X},\mathfrak{R}}.$$

From our previous observations $|\mathfrak{C}_{j,\mathscr{X},\mathfrak{R}}| \geqslant |\mathfrak{R}|$. It is obvious that $|\mathfrak{C}_{j,\mathscr{X},\mathfrak{R}}| = |\mathfrak{R}|$, precisely when $\{\Gamma_{j,\mathscr{X}} : \Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}\} = \mathfrak{D}_{j,\mathscr{X},\mathfrak{R}} \setminus \mathfrak{R}$. Since $\mathfrak{C}_{j,\mathscr{X},\mathfrak{R}} \setminus \mathfrak{R} = \mathfrak{D}_{j,\mathscr{X},\mathfrak{R}} \setminus \mathfrak{R}$, and since $\phi_{j,\mathscr{X}}$ changes the index of a vector only if it is originally j, the profile of a vector space from $\mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}$ changes under the transformation $\phi_{j,\mathscr{X}}$ if the space did not contain $\overline{x}_j$, but its profile contained j. Therefore $\mathfrak{C}_{j,\mathscr{X},\mathfrak{R}}$ is different in profile or cardinality from $\mathfrak{R}$ if and only if $|\mathfrak{R}| < |\mathfrak{C}_{j,\mathscr{X},\mathfrak{R}}|$, or $j \in \underline{p}(U)$ and $\overline{x}_j \notin U$ for some subspace $U \in \Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}$; in other words $j \in \underline{p}(U)$ for some $U \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ such that $\Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}$.

Therefore we define the $(j,\mathscr{X})$-*shift* of $\mathfrak{R}$ as the family $\mathfrak{R}_{j,\mathscr{X}}$, for which

$$\mathfrak{R}_{j,\mathscr{X}} = \begin{cases} \mathfrak{R}, & \text{if } |\mathfrak{C}_{j,\mathscr{X},\mathfrak{R}}| = |\mathfrak{R}| \text{ and for every } \Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}} \\ & U \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma) \text{ implies } j \notin \underline{p}(U), \\ \mathfrak{C}_{j,\mathscr{X},\mathfrak{R}}, & \text{otherwise.} \end{cases}$$

Note that in this case the shift operation might increase (but will never decrease) the cardinality of an intersecting family: more precisely $|\mathfrak{R}| \leqslant |\mathfrak{R}_{j,\mathscr{X}}|$.

We call an intersecting family of chains of subspaces *saturated* when, for every essential basis $\mathscr{X}$ and every integer j such that $1 \leqslant j \leqslant n$, we have $\mathfrak{R} = \mathfrak{R}_{j,\mathscr{X}}$.

We will see that the following properties hold.

Properties.

(1) A $(j,\mathscr{X})$-shift preserves the intersecting property.
(2) A $(j,\mathscr{X})$-shift does not decrease the cardinality of the family (we already know that the shift satisfies this one).
(3) After finitely many shifts an intersecting family becomes saturated.
(4) The intersection of any two chains of a saturated intersecting family contains an initial space.

From properties (1), (2) and (3) it follows that for an intersecting family of k-chains $\mathfrak{R}$ there exists a saturated intersecting family of k-chains $\mathfrak{R}'$ such that $|\mathfrak{R}| \leqslant |\mathfrak{R}'|$. Property (4) is the analogue of a statement that is true for intersecting chains of sets (see [3]).

We call an family of chains of subspaces *standard intersecting* if the intersection of any two chains from the family contains an initial space. If a family contains only one chain, we call it standard intersecting if the chain is nonempty and it contains at least one initial space. Property (4) above essentially states that a saturated family of intersecting k-chains of subspaces is standard intersecting.

It is obvious that $\mathfrak{t}^c_{n,k}[i]$ (resp. $\mathfrak{T}^c_{n,k}[i]$) is a standard intersecting family. Using the fact that an initial segment (resp. initial space) does not change under the shift operation, it is fairly easy to show that this family is compressed (resp. saturated).

5. Existence and structure of saturated intersecting families

Lemma 5.1. *Assume that j is an integer such that $1 \leqslant j \leqslant n$, $\mathscr{X}$ is an essential basis of V, and $\mathfrak{R}$ is an intersecting family of k-chains from $\mathfrak{B}^c_{n,k}$. Then $\mathfrak{R}_{j,\mathscr{X}}$ is also an intersecting family of k-chains from $\mathfrak{B}^c_{n,k}$, and $|\mathfrak{R}_{j,\mathscr{X}}| \geqslant |\mathfrak{R}|$.*

Proof. We will denote $\mathfrak{R}_{j,\mathscr{X}}$ by $\mathfrak{R}'$.
All we need to prove is the intersecting property, *i.e.*, that for all $\Gamma', \Delta' \in \mathfrak{R}'$ we have $|\Gamma' \cap \Delta'| \geqslant 1$. Let $\Gamma, \Delta \in \mathfrak{R}$ be the chains for which Γ' is either Γ or $\Gamma_{j,\mathscr{X}}$ and Δ' is either Δ or $\Delta_{j,\mathscr{X}}$.

Obviously if $\Gamma', \Delta' \in \mathfrak{R}$ (in particular if $\Gamma = \Gamma'$ and $\Delta = \Delta'$), then there is nothing to prove. Also, if $\Gamma' = \Gamma_{j,\mathscr{X}}$ and $\Delta' = \Delta_{j,\mathscr{X}}$, then, since $\Omega = \Delta \cap \Gamma$ is a nonempty chain, $\Omega_{j,\mathscr{X}}$ is also a nonempty chain, and $\Omega_{j,\mathscr{X}} \subseteq \Delta' \cap \Gamma'$.

So, in particular, $|\Gamma' \cap \Delta'| \geqslant 1$ whenever $\Gamma = \Gamma_{j,\mathscr{X}}$ or $\Delta = \Delta_{j,\mathscr{X}}$ or $\Gamma', \Delta' \in \mathfrak{R}$.

So, without loss of generality, assume that $\Gamma \neq \Gamma_{j,\mathscr{X}}$ and $\Delta \neq \Delta_{j,\mathscr{X}}$; moreover, $\Gamma' = \Gamma$ and $\Delta' = \Delta_{j,\mathscr{X}}$ where $\Delta' \notin \mathfrak{R}$. Then from $\Gamma \in \mathfrak{R}'$ there must be a $\Phi \in \mathfrak{R}$ such that $\Phi \neq \Gamma$ and $\Phi_{j,\mathscr{X}} = \Gamma_{j,\mathscr{X}}$. Obviously $\Phi \cap \Gamma = \mathrm{Low}_{j,\mathscr{X}}(\Gamma) \cup \mathrm{High}_{j,\mathscr{X}}(\Gamma)$, and $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma) \neq \emptyset$. Also $(\Phi \setminus \Gamma) \cap \mathrm{Mid}_{j,\mathscr{X}}(\Gamma) = \emptyset$ and $\{L_{j,\mathscr{X}} : L \in \Phi \setminus \Gamma\} = \{L_{j,\mathscr{X}} : L \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)\}$.

Let $U \in \Phi \setminus \Gamma$ such that $\dim U$ is minimal in $\Phi \setminus \Gamma$, and let $U' = U \cap U_{j,\mathscr{X}}$. Then $\dim U' = \dim U - 1$, and by the definition of $U_{j,\mathscr{X}}$ we have that $U \setminus U' = \{u \in U : u = \sum_i \gamma_i x_i \text{ with } \gamma_j \neq 0\}$. Now, by this reasoning, for every element $Y \in \Phi \setminus \Gamma$ we have that $u \in Y$ and $Y = \langle u, (Y \cap Y_{j,\mathscr{X}})\rangle$.

If $\Phi \setminus \Gamma \neq \{L_{j,\mathscr{X}} : L \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)\}$, then, just as before, find a vector w such that every element Z of $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$, $Z = \langle w, (Z \cap Z_{j,\mathscr{X}})\rangle$. Obviously $u \neq w$. Let $U^1 \in \Phi \setminus \Gamma$, $W^1 \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$, and let $U^2 \in \Phi \setminus \Gamma$ be the element with $\dim U^2 = \dim W^1$ and let $W^2 \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ with $\dim W^2 = \dim U^1$. Obviously $U^1 \neq W^2$ and $W^1 \neq U^2$, and from our previous observations $U^2 \cap U^2_{j,\mathscr{X}} = W^1 \cap W^1_{j,\mathscr{X}}$. If $U^1 \subseteq W^1$, then $u \in W^1$. But $u \notin W^1 \cap W^1_{j\mathscr{X}}$; therefore $W^1 = \langle u, (W^1 \cap W^1_{j\mathscr{X}})\rangle = U^2$, which is a contradiction. If we assume that $W^1 \subseteq U^1$, we get a contradiction in a similar way. Therefore, if $\Phi \setminus \Gamma \neq \{L_{j,\mathscr{X}} : L \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)\}$, then the chain Δ can have a nonempty intersection with at most one of $\Phi \setminus \Gamma$ and $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$.

If $\Phi \setminus \Gamma = \{L_{j,\mathscr{X}} : L \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)\}$, then (since every element of $\Phi \setminus \Gamma$ is spanned by $\mathscr{X} \setminus \{x_j\}$, and no element of $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ is spanned by this set), no element of $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ is contained in any element of $\Phi \setminus \Gamma$ by inclusion. Let W be the element of $\Phi \setminus \Gamma$ with the smallest dimension. If there is an element $Y \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ which contains any element of $\Phi \setminus \Gamma$, then it must also contain W. Also, since $W = U_{j,\mathscr{X}}$, then $U_{j,\mathscr{X}} \cap U \subset W$ and $\phi_{j,\mathscr{X}}(u) \in W$. Moreover, $W = \langle \phi_{j,\mathscr{X}}(u), U_{j,\mathscr{X}} \cap U\rangle$. Therefore Y contains both u and $\phi_{j,\mathscr{X}}(u)$. Since $x_j \in \langle u, \phi_{j,\mathscr{X}}(u)\rangle$, we have that $x_j \in Y$, which contradicts the fact that $Y \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$.

So we have that Δ can have a nonempty intersection with at most one of $\Phi \setminus \Gamma$ and $\Gamma \setminus \Phi = \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$.

But since Δ has a nonempty intersection with both Γ and Φ, it follows that $\Omega = \Delta \cap \Phi \cap \Gamma$ must be nonempty. But $\Omega \subset \Delta'$; therefore $|\Delta' \cap \Gamma'| \geqslant 1$. □

We are now ready to prove the following theorem.

Theorem 5.2. *If $\mathfrak{R}$ is an intersecting family of k-chains from $\mathfrak{B}^c_{n,k}$, then there is a saturated intersecting family of k-chains $\mathfrak{R}'$ from $\mathfrak{B}^c_{n,k}$ such that $|\mathfrak{R}| \leqslant |\mathfrak{R}'|$.*

Proof. We define a partial order $\preceq$ on the k-element subsets of $[n]$ as follows: for $I = \{i_1, i_2, \dots, i_k\}$ and $J = \{j_1, j_2, \dots, j_k\}$ (where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$) we have $I \preceq J$ if and only if $i_m \leqslant j_m$ holds for every m.

If $\mathfrak{R}$ is not saturated then there is an essential basis $\mathscr{X}$ of V and an integer j such that either $|\mathfrak{R}_{j,\mathscr{X}}| > |\mathfrak{R}|$, or there is a $\Gamma \in \mathfrak{A}_{j,\mathscr{X},\mathfrak{R}}$ such that $j \in \underline{p}(U)$ for some $U \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$.

In the latter case $\underline{p}(U_{j,\mathscr{X}}) \prec \underline{p}(U)$. However, since the profile of a subspace can not go up in the partial ordering $\preceq$, and there are finitely many subspaces of V, we have that, after finitely many shifts, $\mathfrak{R}$ must become saturated. □

Lemma 5.3. *Let $\mathfrak{R}$ be a saturated family of intersecting k-chains from $\mathfrak{B}^c_{n,k}$. Then $\mathfrak{R}$ is standard intersecting.*

Proof. Assume that $\mathfrak{R}$ is a saturated family of intersecting k-chains that is not standard intersecting. Let $\Gamma, \Delta \in \mathfrak{R}$ such that $\Gamma \cap \Delta$ does not contain any initial spaces. Assume further that amongst the possible choices for the Γ, Δ pairs, Γ minimizes $\sum_{W\in\Gamma} \sum_{l\in\underline{p}(W)} l$.

Since $\mathfrak{R}$ is intersecting, there is some $U \in \Gamma \cap \Delta$ such that, for some positive integers i, j such that $i < j$, we have $j \in \underline{p}(U)$ and $i \notin \underline{p}(U)$. Let $\bar{z} \in U$ be a vector with index j, for $m \neq j$ let $\bar{x}_m$ be an arbitrary vector of index m, and let $\bar{x}_j = \bar{z} - \bar{x}_i$. Then $\mathscr{X} = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ is an essential basis of V, and $U \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$; in particular $\Delta \cap \mathrm{Mid}_{j,\mathscr{X}}(\Gamma) \neq \emptyset$. Since $\mathfrak{R}$ is saturated, we have that $\Gamma' \in \mathfrak{R}$ for $\Gamma' = \Gamma_{j,\mathscr{X}}$. But $\underline{p}(U_{j,\mathscr{X}}) \prec \underline{p}(U)$, and therefore $\sum_{W\in\Gamma'} \sum_{l\in\underline{p}(W)} l < \sum_{W\in\Gamma} \sum_{l\in\underline{p}(W)} l$. It follows from the choice of Γ that $\Gamma' \cap \Delta$ contains an initial space. So the only way this can happen is that $\Gamma' \cap \Delta$ contains some initial space from $\Gamma' \setminus \Gamma$, which means that $\Delta \cap (\Gamma' \setminus \Gamma) \neq \emptyset$, and, as we know, $\Gamma' \setminus \Gamma = \{W_{j,\mathscr{X}} : W \in \mathrm{Mid}_{j,\mathscr{X}}(\Gamma)\}$. However, since Δ is a chain, it intersects at most one of $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma)$ and $\Gamma' \setminus \Gamma$, and $\Delta \cap \mathrm{Mid}_{j,\mathscr{X}}(\Gamma) \neq \emptyset$. We have a contradiction, which means that the statement of our lemma is true. □

6. Technicalities

Lemma 6.1. *For a subspace U and for any integer i, $1 \leqslant i \leqslant n$, we have the following:*

(1) *$V_i \subseteq U$ if and only if $[i] \subseteq \underline{p}(U)$,*
(2) *$V_i = \underline{p}(U)$ if and only if $[i] = \underline{p}(U)$.*

Proof. Obviously $V_i \subseteq U$ implies that $[i] \subseteq \underline{p}(U)$.
We prove the other direction of (1) by induction on i. If $1 \in \underline{p}(U)$, then there is a nonzero vector $\bar{v} \in U$ such that $\bar{v} \in V_1$. But then $\langle \bar{v} \rangle = V_1$, so $V_1 \subseteq U$.

Assume that $[i-1] \subseteq \underline{p}(U)$ implies $V_{i-1} \subseteq U$. Assume further that $[i] \subseteq \underline{p}(U)$. Then

$V_{i-1} \subseteq U$ and there is $\overline{v} \in V_i \setminus V_{i-1}$ such that $\overline{v} \in U$. But then $\langle V_{i-1}, \overline{v} \rangle \subseteq V_i$ and $\dim(\langle V_{i-1}, \overline{v} \rangle) = i = \dim(V_i)$, from which $V_i \subseteq U$ follows.

(2) is obvious from (1) and the fact that $|\underline{p}(U)| = \dim(U)$. □

Lemma 6.2. *If U is a subspace of V and $j \notin \underline{p}(U)$, then*

$$\left|\left\{U' \ : \ U' = \langle U \cup \{\overline{u}\} \rangle \quad \textit{for some} \quad \overline{u} \in \mathscr{W}_j \right\}\right| = q^{\left|[j] \setminus (\underline{p}(U) \cup \{j\})\right|}.$$

Proof. Define a relation $\sim$ as follows: for $\overline{v}, \overline{w} \in \mathscr{W}_j$, $\overline{v} \sim \overline{w}$ if and only if $\langle U \cup \{\overline{v}\} \rangle = \langle U \cup \{\overline{w}\} \rangle$. Obviously $\sim$ is an equivalence relation on $\mathscr{W}_j$; moreover, $|\{U' : U' = \langle U \cup \{\overline{u}\} \rangle$ for some $\overline{u} \in \mathscr{W}_j\}|$ is the number of equivalence classes of $\sim$ on $\mathscr{W}_j$. Clearly $|\mathscr{W}_j| = (q-1)q^{j-1}$.

Note that from $j \notin \underline{p}(U)$ we have $U \cap V_j = (U \cap V_{j-1}) \cup (U \cap \mathscr{W}_j) = U \cap V_{j-1}$. Assume therefore that $\overline{u}$ is a fixed vector from $\mathscr{W}_j$. We want to compute the number of vectors in $\mathscr{W}_j$ that give the same span added to U as $\overline{u}$ gives, *i.e.*, $|\{\overline{v} : \overline{v} \sim \overline{u}\}|$.

Let $\overline{y}$ be an arbitrary element of V. Using an essential basis $\chi = \{\overline{x}_1, \ldots, \overline{x}_n\}$ where $\overline{x}_j = \overline{u}$, the following are easy to see: $I(\overline{y}) = j$ if and only if, for some nonzero scalar α and some $\overline{w} \in V_{j-1}$, we have $\overline{y} = \alpha\overline{u} + \overline{w}$. Also, if $\overline{y} = \beta\overline{u} + \overline{w}$ for some constant β and vector $\overline{w}$ with $I(\overline{w}) \geqslant j$, then $I(\overline{y}) = I(\overline{w})$.

From these and the fact that $U \cap \mathscr{W}_j = \emptyset$, it is obvious that $= \overline{u} \sim \overline{v}$ if and only if $\overline{v} = \alpha_{\overline{v}}\overline{u} + \overline{w}_{\overline{v}}$ for some nonzero scalar $\alpha_{\overline{v}} \in GF(q)$ and $\overline{w}_{\overline{u}} \in U \cap V_{j-1} = U \cap V_j$; moreover, if $\overline{v}_1 \neq \overline{v}_2$ such that $\overline{v}_1 \sim \overline{u} \sim \overline{v}_2$, then $\alpha_{\overline{v}_1} \neq \alpha_{\overline{v}_2}$ or $\overline{w}_{\overline{v}_1} \neq \overline{w}_{\overline{u}_2}$. We have $q-1$ choices for α and $q^{\dim(U \cap V_j)}$ choices for $\overline{w}$: therefore $|\{\overline{v} : \overline{v} \sim \overline{u}\}| = (q-1)q^{\dim(U \cap V_j)}$.

So,

$$\left|\left\{U' \ : \ U' = \langle U \cup \{\overline{u}\} \rangle \quad \text{for some} \quad \overline{u} \in \mathscr{W}_j \right\}\right| = q^{j-1-\dim(U \cap V_j)}.$$

From the fact that, for an essential basis $\{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k\}$ of U, we have that $\underline{p}(U) = \{I(\overline{v}_1), I(\overline{v}_2), \ldots, I(\overline{v}_k)\}$, we have that $q^{j-1-\dim(U \cap V_j)} = q^{\left|[j] \setminus (\underline{p}(U) \cup \{j\})\right|}$. □

From this it is easy to prove the following.

Lemma 6.3. *Let $\underline{\gamma}$ be a chain in $\mathfrak{b}^c_{n,l}$. Then the number of chains from $\mathfrak{B}^c_{n,l}$ with profile $\underline{\gamma}$ is $q^{\mathfrak{g}(\underline{\gamma})}$.*

Proof. Let $\underline{\gamma} = \{\underline{m}^1 \subset \underline{m}^2 \subset \cdots \subset \underline{m}^l\}$. Note that

$$\sum_{p=1}^{l} \sum_{r \in \underline{m}^p \setminus \underline{m}^{p-1}} \left|[r] \setminus \underline{m}^p\right| = \sum_{p=1}^{l} \sum_{r \in \underline{m}^p \setminus \underline{m}^{p-1}} \left|[r]\right| - \left|\underline{m}^p \cap [r]\right|.$$

For every $p : 0 \leqslant p \leqslant l$ we define $\underline{\gamma}_p$ as $(\underline{m}^1 \subset \cdots \subset \underline{m}^p)$. We prove that the number of subspaces with profile $\underline{\gamma}_p$ is $q^{\mathfrak{g}(\underline{\gamma}_p)}$ by induction on p.

For $p = 0$ the statement is trivial. For $p = 1$ the statement follows from Lemma 6.2 by induction on $|\underline{m}^1|$ as follows. Let $\underline{m}^1 = \{i_1 < i_2 < \cdots < i_s\}$. If $s = 0$ then the gap is 0, and the only subspace with profile $\underline{m}^1$ is the zero space.

Let a be an integer such that $1 \leqslant a \leqslant s$ and assume that we know the statement for every b such that $b < a$. The number of subspaces with profile $\{i_1 < i_2 < \cdots < i_a\}$ equals the number of subspaces with profile $\{i_1 < i_2 < \cdots < i_{a-1}\}$ times the number of ways we can extend such a subspace to a subspace with a vector with index i_a, which, by the induction hypothesis and Lemma 6.2, is $q^{\mathfrak{g}(\{i_1,i_2,\ldots,i_{a-1}\})}q^{i_a-1-(a-1)} = q^{\mathfrak{g}(\{i_1,i_2,\cdots,i_a\})}$.

Assume we know the statement for $p' < p$. Then, since the number of chains of subspaces with profile $\underline{\gamma}_p$ is the same as the number of chains of subspaces with profile $\underline{\gamma}_{p-1}$ times the number of ways we can extend a space with profile $\underline{m}^{p-1}$ to a space with profile $\underline{m}^p$, as before we have that the number of chains of subspaces with profile $\underline{\gamma}$ is

$$q^{\mathfrak{g}(\underline{\gamma}_{p-1})+\sum_{r\in\underline{m}^p\setminus\underline{m}^{p-1}}\left|[r]\right|-\left|\underline{m}^p\cap[r]\right|} = q^{\mathfrak{g}(\underline{\gamma}_p)}. \qquad \square$$

Note that Lemma 6.1 is an easy corollary of Lemma 6.3.

7. Correspondence between intersecting *k*-chains of subspaces and intersecting *k*-chains of subsets

Theorem 7.1. *Let $\mathfrak{R}$ be a standard intersecting family of k-chains from $\mathfrak{B}^c_{n,k}$ and let $\mathfrak{r}$ be a standard intersecting family of k-chains from $\mathfrak{b}^c_{n,k}$. The following hold.*

(1) *$\mathfrak{H}_{\mathfrak{r}}$ is a standard intersecting family of k-chains from $\mathfrak{B}^c_{n,k}$, and $\mathfrak{h}_{\mathfrak{H}_{\mathfrak{r}}} = \mathfrak{r}$.*
(2) *$\mathfrak{h}_{\mathfrak{R}}$ is a standard intersecting family of k-chains from $\mathfrak{b}^c_{n,k}$.*
(3) *Let $\underline{\gamma} \in \mathfrak{h}_{\mathfrak{R}}$, and let Γ be an arbitrary k-chain of subspaces such that $\underline{\pi}(\Gamma) = \underline{\gamma}$. Then $\mathfrak{R} \cup \{\Gamma\}$ is also a standard family of intersecting k-chains.*
(4) *If $\mathfrak{R}$ is a maximal standard intersecting family* (i.e., *for any $\Gamma \notin \mathfrak{R}$, $\{\Gamma\} \cup \mathfrak{R}$ is not standard intersecting), then $\mathfrak{H}_{\mathfrak{h}_{\mathfrak{R}}} = \mathfrak{R}$. Moreover, $\mathfrak{h}_{\mathfrak{R}}$ is a maximal standard intersecting family of k-chains.*
(5) *If $\mathfrak{r}$ is a maximal standard intersecting family, then it is compressed.*

Proof. From Lemma 5.3 it follows that the intersection of two k-chains contains some initial space V_i if and only if the intersection of their profiles contains the initial segment $[i]$.

From this (1) follows, since if Γ, Δ are chains from an $\mathfrak{H}_{\mathfrak{r}}$, then the intersection of their profiles contains some initial segment, so their intersection contains an initial space, and therefore $\mathfrak{H}_{\mathfrak{r}}$ is standard intersecting. The rest is obvious.

The proof of (2) is similar to the proof of (1).

For (3) consider the following. If $\underline{\gamma} \in \mathfrak{h}_{\mathfrak{R}}$ then there is a $\Gamma' \in \mathfrak{R}$ such that $\underline{\pi}(\Gamma') = \underline{\gamma}$. If Γ is a chain such that $\underline{\pi}(\Gamma) = \underline{\gamma}$, then for any integer $i : 0 \leqslant i \leqslant n$ we have $V_i \in \Gamma'$ if and only if $[i] \in \underline{\gamma}$ if and only if $V_i \in \Gamma$. Since Γ' intersects every element of $\mathfrak{R}$ in some initial space, we have that Γ also intersects every element of $\mathfrak{R}$ in some initial space, and therefore $\mathfrak{R} \cup \{\Gamma\}$ is also an intersecting family of k-chains.

(4) is a consequence of (3).

For (5) notice that an initial segment does not move under the shift operation. Therefore if $\underline{\gamma} \in \mathfrak{r}$, and $\underline{\gamma}'$ is some shift of $\underline{\gamma}$, then $\underline{\gamma}'$ contains all the initial segments from $\underline{\gamma}$, and therefore it must intersect any chain from $\mathfrak{r}$ in some initial segment. So $\underline{\gamma}' \in \mathfrak{r}$ from the maximality of $\mathfrak{r}$. □

Theorem 7.2. *Let $\mathfrak{R}$ be a saturated intersecting family of k-chains from $\mathfrak{B}^c_{n,k}$, and $\mathfrak{r}$ a compressed intersecting family of k-chains from $\mathfrak{b}^c_{n,k}$. The following hold.*

(1) $\mathfrak{h}_{\mathfrak{R}}$ *is a compressed intersecting family.*
(2) $\mathfrak{H}_{\mathfrak{r}}$ *is a saturated intersecting family.*
(3) *If $\mathfrak{r}$ is a maximal compressed intersecting family, then $\mathfrak{H}_{\mathfrak{r}}$ is a maximal saturated intersecting family.*
(4) *If $\mathfrak{R}$ is a maximal saturated intersecting family, then $\mathfrak{h}_{\mathfrak{R}}$ is a maximal compressed intersecting family.*

Proof. Assume that (1) does not hold, that is, $\mathfrak{h}_{\mathfrak{R}}$ is not compressed. Then there are positive integers i, j, $i < j$, such that, for some $\underline{\gamma} \in \mathfrak{h}_{\mathfrak{R}}$ the (i, j)-shift of $\underline{\gamma}$ is not in $\mathfrak{h}_{\mathfrak{R}}$. Let us denote the (i, j)-shift of $\underline{\gamma}$ by $\underline{\gamma}'$. Let $\underline{\gamma} = (\underline{l}^1 \subset \underline{l}^2 \subset \cdots \subset \underline{l}^k)$ and $\underline{\gamma}' = (\underline{m}^1 \subset \underline{m}^2 \subset \cdots \subset \underline{m}^k)$. There is some $r_1 : 1 \leqslant r_1 \leqslant k$ such that $j \in \underline{l}^s$ if and only if $s \geqslant r_1$, and there is some r_2, $1 \leqslant r_2 \leqslant k + 1$ such that $i \in \underline{l}^s$ if and only if $s \geqslant r_2$. From the fact that $\underline{\gamma}' \neq \underline{\gamma}$ we have that $r_1 < r_2$.

Consider a $\Gamma \in \mathfrak{R}$ such that $\underline{\pi}(\Gamma) = \underline{\gamma}$, and assume that $\Gamma = (U^1 \subset U^2 \subset \cdots \subset U^k$. Obviously $\underline{p}(U^s) = \underline{l}^s$ for every s, $1 \leqslant s \leqslant k$. Let $\overline{y} \in U^{r_1}$ such that $I(\overline{y}) = j$. If $r_2 < k + 1$, then let $\overline{x}_i$ be a vector from U^{r_2} such that $I(\overline{x}_i) = i$; otherwise let $\overline{x}_i$ be any vector with $I(\overline{x}_i) = i$. Let $\overline{x}_j = \overline{y} - \overline{x}_i$: obviously $I(\overline{x}_j) = j$. For $m \neq i, j$ let $\overline{x}_m$ be a vector such that $I(\overline{x}_m) = m$. Let $\mathscr{X} = \{\overline{x}_1, \ldots, \overline{x}_n\}$. Since if $s \geqslant r_2$ then $\overline{y}, \overline{x}_i \in U^s$, therefore $\overline{x}_j \in U^s$, from which it follows that $U^s_{j,\mathscr{X}} = U^s$. If $s < r_1$, then $j \notin \underline{p}(U^s)$; therefore $\underline{p}(U^s) = \underline{p}(U^s_{j,\mathscr{X}})$. If $r_1 \leqslant s < r_2$, then $\overline{y} \in U^s$ and $i \notin \underline{p}(U^s)$; therefore (since $\phi_{j,\mathscr{X}}(\overline{y}) = \overline{x}_i$) we have that $\underline{p}(U^s_{j,\mathscr{X}}) = \big(\underline{p}(U^s) \setminus \{j\}\big) \cup \{i\}$. From these $\underline{\pi}(\Gamma_{j,\mathscr{X}}) = \underline{\gamma}'$. But since $\mathfrak{R}$ was compressed, $\Gamma_{j,\mathscr{X}} \in \mathfrak{R}$, from which $\underline{\gamma}' \in \mathfrak{h}_{\mathfrak{R}}$ follows. This is a contradiction; therefore $\mathfrak{h}_{\mathfrak{R}}$ is compressed, and we have concluded (1).

Assume that (2) does not hold, that is, $\mathfrak{H}_{\mathfrak{r}}$ is not saturated. Then there is a $\Gamma \in \mathfrak{H}_{\mathfrak{r}}$ such that, for some $\mathscr{X} = \{\overline{x}_1, \ldots, \overline{x}_n\}$ essential basis of V and some integer j, the chain $\Gamma_{j,\mathscr{X}}$ is not in $\mathfrak{H}_{\mathfrak{r}}$. Let us denote $\underline{\pi}(\Gamma)$ by $\underline{\gamma}$, and $\underline{\pi}(\Gamma_{j,\mathscr{X}})$ by $\underline{\gamma}'$. By the definition of $\mathfrak{H}_{\mathfrak{r}}$ we have that $\gamma' \notin \mathfrak{r}$. Consider $\mathrm{Mid}_{j,\mathscr{X}}(\Gamma) = (U^1 \subset U^2 \subset \cdots \subset U^l)$, and let m_1 be an integer such that $j \in \underline{p}(U^s)$ if and only if $s \geqslant m_1$. From the fact that $\underline{\gamma}' \neq \underline{\gamma}$ it follows that $1 \leqslant m_1 \leqslant l$. Let i_1 be the integer such that $\{i_1\} = \underline{p}(U^{m_1}_{j,\mathscr{X}}) \setminus \underline{p}(U^m_1)$ (the existence of such an integer follows from the fact that $\dim(U^m_1 \cap U^{m_1}_{j,\mathscr{X}}) = \dim(U^{m_1}) - 1$). Obviously $i_1 < j$. Let us denote the (i_1, j)-shift of $\underline{\gamma}$ by $\underline{\gamma}_1$. Since $\mathfrak{r}$ was compressed, we have that $\gamma_1 \in \mathfrak{r}$. If $\underline{\gamma}_1 = \underline{\gamma}'$, then we have a contradiction. Therefore $\underline{\gamma}' \neq \underline{\gamma}_1$. But this is only possible if there is some m_2, $m_1 < m_2 \leqslant l$ such that $i_1 \in \underline{p}(U^s)$ if and only if $s \geqslant m_2$. Let $\{i_2\} = \underline{p}(U^{m_2}_{j,\mathscr{X}}) \setminus \underline{p}(U^{m_2})$, and it follows as before that $i_2 < i_1$. Let $\underline{\gamma}_2$ be the (j, i_2)-shift of $\underline{\gamma}_1$. As before, $\underline{\gamma}_2 \in \mathfrak{r}$. If $\gamma' = \gamma_2$, then we have a contradiction. So assume we have defined $m_1 < m_2 < \cdots < m_r \leqslant l$ and $i_1 > i_2 > \cdots > i_r \geqslant 1$, and a sequence of chains $\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_r$ such that $i_d \in \underline{p}(U^s)$ and $i_{d+1} \in \underline{p}(U^s_{j,\mathscr{X}})$ if and only if $s \geqslant m_{d+1}$ (note that we can use $i_0 = j$); also $\underline{\gamma}_{a+1} \in \mathfrak{r}$ is the (j, i_{a+1})-shift of $\underline{\gamma}_a$ for every a, $1 \leqslant a \leqslant r - 1$, and moreover $\underline{\gamma}' \notin \{\underline{\gamma}_1, \ldots, \underline{\gamma}_{r-1}\}$. If $\underline{\gamma}' = \gamma_r$, then we have a contradiction, since $\underline{\gamma}' \in \mathfrak{r}$, otherwise we can define $m_{r+1}, i_{r+1}, \underline{\gamma}_{r+1}$ in a similar fashion. Since neither of these sequences can continue indefinitely, sooner or later we have to conclude $\underline{\gamma}' \in \mathfrak{r}$, which is a contradiction. Therefore we have proved (2).

Assume that (3) does not hold, that is, $\mathfrak{H}_{\mathfrak{r}}$ is not maximal. Then there is some $\Gamma \notin \mathfrak{H}_{\mathfrak{r}}$ such that $\{\Gamma\} \cup \mathfrak{H}_{\mathfrak{r}}$ is intersecting. From the definition of $\mathfrak{H}_{\mathfrak{r}}$ we have that $\underline{\pi}(\Gamma) \notin \mathfrak{r}$. But since $\mathfrak{H}_{\mathfrak{r}} \cup \{\Gamma\}$ is intersecting, we have that $\mathfrak{r} \cup \{\underline{\pi}(\Gamma)\}$ is also intersecting, which contradicts the maximality of $\mathfrak{r}$. Therefore $\mathfrak{H}_{\mathfrak{r}}$ is maximal, and we have concluded (3).

Assume that (4) does not hold, that is, $\mathfrak{h}_{\mathfrak{R}}$ is not maximal. Then there is some $\underline{\gamma} \in \mathfrak{b}^c_{n,k} \cap \mathfrak{h}_{\mathfrak{R}}$ such that $\mathfrak{r}_1 = \mathfrak{h}_{\mathfrak{R}} \cup \{\underline{\gamma}\}$ is intersecting. Let $\mathfrak{r}_2$ be the compressed version of $\mathfrak{r}_1$, that is, what we obtained from $\mathfrak{r}_1$ after those finitely many shifts that make it compressed. From the fact that $\mathfrak{h}_{\mathfrak{R}}$ was compressed it follows that $\mathfrak{h}_{\mathfrak{R}} \subset \mathfrak{r}_2$; also $\mathfrak{H}_{\mathfrak{r}_2}$ is intersecting and $\mathfrak{R} \subset \mathfrak{H}_{\mathfrak{r}_2}$. But this contradicts the maximality of $\mathfrak{R}$: therefore we have concluded (4). □

Corollary 7.3. *If $\mathfrak{R}$ is a maximal standard intersecting family of k-chains, then $\mathfrak{R}$ is saturated.*

Proof. By Theorem 7.1 we have that $\mathfrak{h}_{\mathfrak{R}}$ is a compressed intersecting family of k-chains and $\mathfrak{R} = \mathfrak{H}_{\mathfrak{h}_{\mathfrak{R}}}$. It follows from Theorem 7.2 that $\mathfrak{R} = \mathfrak{H}_{\mathfrak{h}_{\mathfrak{R}}}$ is saturated. □

So we know that the maximal compressed intersecting k-chains from $\mathfrak{b}^c_{n,k}$ and the maximal saturated intersecting k-chains from $\mathfrak{B}^c_{n,k}$ correspond to each other; in particular, the maximal standard intersecting k-chains of subspaces and subsets correspond to each other. Therefore, by Theorem 5.2, in order to find an intersecting k-chain with maximum cardinality, it is enough to consider the families $\mathfrak{H}_{\mathfrak{r}}$, where $\mathfrak{r}$ is a maximal standard intersecting k-chain, or even more specifically a maximal compressed intersecting k-chain.

8. The $\widehat{\ }$ operation and standard intersecting families

Lemma 8.1. *Let n, c, k be nonnegative integers such that $1 \leqslant k \leqslant n - 2c + 1$. Assume that $\underline{\gamma} \in \mathfrak{b}^c_{n,k}$ where $\underline{\gamma} = (\underline{m}^1 \subset \underline{m}^2 \subset \cdots \subset \underline{m}^k)$. Define $\underline{m}^{k+1} = [n]$ and $\underline{m}^0 = \emptyset$. Let i be the integer such that $0 \leqslant i \leqslant k+1$ and $n - c \notin \underline{m}_j$ if and only if $j < i$ and $j \geqslant 1$. Then*

$$\mathfrak{g}(\underline{\gamma}) = \mathfrak{g}(\widehat{\underline{\gamma}}) + \left(n - c - \left|\underline{m}^i \cap [n-c]\right|\right) + \left|\underline{m}^{i-1} \setminus [n-c]\right|.$$

Proof. If $i = k+1$ or $i = 0$ or $\underline{m}^i \neq \underline{m}^{i-1} \cup \{n-c\}$, then $\widehat{\underline{\gamma}} \in \mathfrak{b}^0_{\widehat{n,k}}$. By the definition of the gap

$$\begin{aligned} \mathfrak{g}(\widehat{\underline{\gamma}}) &= \sum_{p=1}^{k} \sum_{r \in \widehat{\underline{m}^p} \setminus \widehat{\underline{m}^{p-1}}} \left|[\widehat{r}] \setminus \widehat{\underline{m}^p}\right| \\ &= \sum_{p=1}^{k} \sum_{r \in [n-c-1] \cap \left(\underline{m}^p \setminus \underline{m}^{p-1}\right)} \left|[r] \setminus \underline{m}^p\right| \\ &\quad + \sum_{p=1}^{k} \sum_{r \in \underline{m}^p \setminus \left(\underline{m}^{p-1} \cup [n-c]\right)} \left|\left([r] \setminus \{n-c\}\right) \setminus \left(\underline{m}^p \setminus \{n-c\}\right)\right| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=1}^{k} \sum_{r\in \underline{m}^p\setminus \underline{m}^{p-1}} \big|[r]\setminus \underline{m}^p\big| \\
&\qquad - \big|[n-c]\setminus \underline{m}^i\big| - \sum_{p=1}^{i-1} \sum_{r\in \underline{m}^p\setminus\left(\underline{m}^{p-1}\cup[n-c]\right)} 1 \\
&= \mathfrak{g}(\underline{\gamma}) - \big(n-c-\big|\underline{m}^i\cap[n-c]\big|\big) - \sum_{p=1}^{i-1}\big|\underline{m}^p\setminus\left(\underline{m}^{p-1}\cup[n-c]\right)\big|.
\end{aligned}
$$

Since the sets involved in the summation above are disjoint,

$$
\begin{aligned}
\mathfrak{g}(\widehat{\underline{\gamma}}) &= \mathfrak{g}(\underline{\gamma}) - \big(n-c-\big|\underline{m}^i\cap[n-c]\big|\big) - \Big|\bigcup_{p=1}^{i-1}\left(\underline{m}^p\setminus(\underline{m}^{p-1}\cup[n-c])\right)\Big| \\
&= \mathfrak{g}(\underline{\gamma}) - \big(n-c-\big|\underline{m}^i\cap[n-c]\big|\big) - \big|\underline{m}^{i-1}\setminus[n-c])\big|.
\end{aligned}
$$

If $0<i\leqslant k$ and $\underline{m}^i=\underline{m}^{i-1}\cup\{n-c\}$, then $\widehat{\underline{\gamma}}\in\mathfrak{b}^c_{\widehat{n,k-1}}$. In a similar way to the previous case,

$$
\begin{aligned}
\mathfrak{g}(\widehat{\underline{\gamma}}) &= \sum_{p=1}^{i-1}\sum_{r\in\widehat{\underline{m}^p}\setminus\widehat{\underline{m}^{p-1}}}\big|[\widehat{r}]\setminus\widehat{\underline{m}^p}\big| + \sum_{p=i+1}^{k}\sum_{r\in\widehat{\underline{m}^p}\setminus\widehat{\underline{m}^{p-1}}}\big|[\widehat{r}]\setminus\widehat{\underline{m}^p}\big| \\
&= \sum_{p=1}^{i-1}\sum_{r\in[n-c-1]\cap\left(\underline{m}^p\setminus\underline{m}^{p-1}\right)}\big|[r]\setminus\underline{m}^p\big| \\
&\qquad + \sum_{p=1}^{i-1}\sum_{r\in\underline{m}^p\setminus\left(\underline{m}^{p-1}\cup[n-c]\right)}\big|\left([r]\setminus\{n-c\}\right)\setminus\left(\underline{m}^p\setminus\{n-c\}\right)\big| \\
&\qquad + \sum_{p=i+1}^{k}\sum_{r\in\underline{m}^p\setminus\underline{m}^{p-1}}\big|[r]\setminus\underline{m}^p\big| \\
&= \sum_{p=1}^{k}\sum_{r\in\underline{m}^p\setminus\underline{m}^{p-1}}\big|[r]\setminus\underline{m}^p\cap[r]\big| \\
&\qquad - \sum_{r\in\underline{m}^i\setminus\underline{m}^{i-1}}\big|[r]\setminus\underline{m}^i\cap[r]\big| - \sum_{p=1}^{i-1}\big|\underline{m}^p\setminus\left(\underline{m}^{p-1}\cup[n-c]\right)\big| \\
&= \mathfrak{g}(\underline{\gamma}) - \big(n-c-\big|\underline{m}^i\cap[n-c]\big|\big) - \Big|\bigcup_{p=1}^{i-1}\left(\underline{m}^p\setminus(\underline{m}^{p-1}\cup[n-c])\right)\Big| \\
&= \mathfrak{g}(\underline{\gamma}) - \big(n-c-\big|\underline{m}^i\cap[n-c]\big|\big) - \big|\underline{m}^{i-1}\setminus[n-c])\big|. \qquad\square
\end{aligned}
$$

Lemma 8.2. *Let n,k,c be nonnegative integers such that $1\leqslant k\leqslant n-2c+1$. Let $\mathfrak{r}\subseteq\mathfrak{b}^c_{n,k}$ be a standard intersecting family, and let $\mathfrak{r}'=\{(\underline{l}^1\subset\cdots\subset\underline{l}^k)\in\mathfrak{r} : \underline{l}^k=[n-c]\}$. The following hold.*

(1) *If* $\mathfrak{r}$ *is compressed, then so is* $\mathfrak{r}'$.
(2) *If there is a* $\underline{\gamma}^* = (\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r}$ *such that* $n - c \in \underline{l}^1$, *then* $\mathfrak{r} \subseteq \mathfrak{t}^c_{n,k}[n-c]$. *In particular* $\mathfrak{r} = \mathfrak{r}'$.

Proof. (1) follows from the fact that an initial segment (in particular $[n-c]$) does not change under an (i,j)-shift. Therefore, if $\mathfrak{r}$ is compressed and $\underline{\gamma} \in \mathfrak{r}'$, then any shift $\underline{\gamma}'$ of the chain γ is also in $\mathfrak{r}$; therefore we can conclude that $\underline{\gamma}'$ is also in $\mathfrak{r}'$

For (2) assume there is a $\underline{\gamma}^* = (\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r}$ such that $n - c \in \underline{l}^1$. Since $\mathfrak{r}$ is standard intersecting, $\underline{\gamma}^*$ intersects every $\underline{\gamma} \in \mathfrak{r}$ in at least one initial segment. In particular, $\underline{\gamma}^*$ contains at least one initial segment. But from the fact that $n - c \in \underline{l}^1$ and $|\underline{l}^k| \leqslant n - c$ it follows that the set $\{\underline{l}^1, \ldots, \underline{l}^k\}$ can contain at most one initial segment (and if it in fact contains one, then it must be $\underline{l}^k = [n-c]$). Since every other element of $\mathfrak{r}$ must intersect $\underline{\gamma}^*$ in this initial segment, we can deduce that $\mathfrak{r} \subseteq \mathfrak{t}^c_{n,k}[n-c]$. We conclude (2) □

Lemma 8.3. *Let* n, k, c *be nonnegative integers such that* $1 < k \leqslant n - 2c + 1$. *Let* $\mathfrak{r} \subseteq \mathfrak{b}^c_{n,k}$ *be a standard intersecting family of* k*-chains. Let* $\mathfrak{r}'$ *be defined as in Lemma 8.2. If* $\mathfrak{r} \neq \mathfrak{r}'$, *then the following are true.*

(1) *For every* $(\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r}$ *we have that* $n - c \notin \underline{l}^1$.
(2) *The following sets partition* $\mathfrak{r}$:

$$\begin{aligned}
\mathfrak{r}^c_{n,k}(i) &= \{(\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r} \ : \underline{l}^{i+1} = \underline{l}^i \cup \{n-c\}\}, \qquad \textit{for } 1 \leqslant i \leqslant k-1, \\
\mathfrak{r}^c_{n,k}(k) &= \{(\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r} \ : n - c \notin \underline{l}^k,\ |\underline{l}^k| = n - c\}, \\
\mathfrak{r}^c_{n,k}(0) &= \mathfrak{r} \setminus \bigcup_{i=1}^{k} \mathfrak{r}^c_{n,k}(i).
\end{aligned}$$

(3) $\widehat{\mathfrak{r}^c_{n,k}(i)} \subseteq \mathfrak{b}^c_{\widehat{n,k-1}}$ *is a standard intersecting family for every* $1 \leqslant i \leqslant k-1$.
(4) $\widehat{\mathfrak{r}^c_{n,k}(k)} \subseteq \mathfrak{b}^0_{\widehat{n,k}}$ *is a standard intersecting family of* k*-chains such that the first element of every chain has cardinality at least* c, *and the last element of every chain has cardinality* $n - c$.
(5) $\widehat{\mathfrak{r}^c_{n,k}(0)} \subseteq \mathfrak{b}^c_{\widehat{n,k}}$ *is a standard intersecting family.*
(6) $\widehat{\ }$ *is a bijection from* $\mathfrak{r}^c_{n,k}(i)$ *to* $\widehat{\mathfrak{r}^c_{n,k}(i)}$ *for every* i, $1 \leqslant i \leqslant k$.
(7) *if* $c = 0$, *then* $\mathfrak{r}^c_{n,k}(k) = \emptyset$.

Proof. (1) follows from Lemma 8.2, and (2) follows from (1).
Obviously $\widehat{\mathfrak{r}^c_{n,k}(i)} \subseteq \mathfrak{b}^c_{\widehat{n,k-1}}$ for every $1 \leqslant i \leqslant k-1$. For a given i, $1 \leqslant i \leqslant k-1$, every two chains from $\mathfrak{r}^c_{n,k}(i)$ intersect in at least one initial segment. But then two chains from $\widehat{\mathfrak{r}^c_{n,k}(i)}$ must also intersect in at least one initial segment (since the initial segments they contain are in the form $[i]$ for some $1 \leqslant i \leqslant n - c$, and initial segments map onto initial segments under $\widehat{\ }$). Therefore $\widehat{\mathfrak{r}^c_{n,k}(i)}$ is standard intersecting for $1 \leqslant i \leqslant k-1$. So we have (3).

(4) follows from the definition of $\mathfrak{r}^c_{n,k}(k)$.

It is easy to see that

$$\begin{aligned}\mathfrak{r}^c_{n,k}(0) &= \{(\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r} \;:\; n-c \notin \underline{l}^k,\ |\underline{l}^k| \leqslant n-1-c\} \\ &\qquad \cup \{(\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r} \;:\; \{n-c\} \subset \underline{l}^{i+1} \setminus \underline{l}^i \text{ for some } i,\ 1 \leqslant i < k\},\end{aligned}$$

Therefore $\widehat{\mathfrak{r}^c_{n,k}(0)} \subseteq \mathfrak{b}^c_{\widehat{n,k}}$ is a standard intersecting family. We have concluded (5).

(6) follows from the definitions of $\mathfrak{r}^c_{n,k}(i)$ and $\widehat{\ }$.

(7) is a consequence of the fact that if, for an $\underline{l} \subset [n]$, we have $|\underline{l}| = n$, then $\underline{l} = [n]$, in particular $n \in \underline{l}$. □

Note that the above lemmas can be proven similarly for t-intersecting families of k-chains (when the intersection of two chains is of cardinality at least t), with the only exception of the chains in (3) being only standard $\max(1, t-1)$-intersecting.

Lemma 8.4. *Let n, c be positive integers such that $2c+1 \leqslant n$. Let $\underline{a}, \underline{b} \subset [n]$ such that $n-c \geqslant |\underline{a}| \geqslant |\underline{b}|$, $|\underline{a} \setminus [n-c]| \leqslant |\underline{b} \setminus [n-c]|$ and $|\underline{a} \cap [n-c]| \geqslant |\underline{b} \cap [n-c]|$. Let $\underline{\alpha}^* = \{\underline{c} \subseteq [n] \;:\; \underline{a} \subset \underline{c},\ |\underline{c}| = n-c+1\}$ and $\underline{\beta}^* = \{\underline{c} \subseteq [n] \;:\; \underline{b} \subset \underline{c},\ |\underline{c}| = n-c+1\}$. There is an injection $\chi : \underline{\alpha}^* \to \underline{\beta}^*$ such that, if $\chi(\underline{c}_1) = \underline{c}_2$, then $|\underline{c}_1 \cap [n-c] \geqslant |\underline{c}_2 \cap [n-c]|$ and $|\underline{c}_1 \setminus [n-c]| \leqslant |\underline{c}_2 \setminus [n-c]|$. Moreover, χ is a bijection if and only if $|\underline{a}| = |\underline{b}|$ or $c = 1$.*

Proof. If $c = 1$, then $\underline{\beta}^* = \underline{\alpha}^* = \{[n]\}$, so the statement is trivial.
Assume therefore that $c > 1$. Let $|\underline{a}| = a$ and $|\underline{b}| = b$. We have $b \leqslant a \leqslant n-c$. If $a = b$, then obviously $|\underline{\alpha}^*| = |\underline{\beta}^*|$, and if $a > b$, then

$$\begin{aligned}\frac{|\underline{\alpha}^*|}{|\underline{\beta}^*|} &= \frac{\binom{n-a}{n-c+1-a}}{\binom{n-b}{n-c+1-b}} = \frac{(n-a)!(n-c+1-b)!}{(n-b)!(n-c+1-a)!} \\ &= \frac{(n-c-b+1)}{(n-b)}\frac{(n-c-b)}{(n-b-1)}\cdots\frac{(n-c-a+2)}{(n-a+1)} < 1.\end{aligned}$$

Therefore, if for a $c > 1$ we have an injection between the two families, then it is a bijection precisely when $a = b$.

We will prove the existence of such an injection by induction on $s = |\underline{a} \cap [n-c]| - |\underline{b} \cap [n-c]|$. Without loss of generality we can assume that $b = a$. (Otherwise replace $\underline{b}$ with a $\underline{b}' \subset [n]$ for which $\underline{b} \subset \underline{b}'$, $|\underline{b}'| = |\underline{a}|$, and $\underline{b}' \setminus [n-c] = \underline{b} \setminus [n-c]$.) Since a permutation on the elements of $[n-c]$ and $[n] \setminus [n-c]$ does not change the properties we look for, we can also assume that $\underline{b} \cap [n-c] \subseteq \underline{a} \cap [n-c]$ and $\underline{a} \setminus [n-c] \subseteq \underline{b} \setminus [n-c]$. Also, we can remove the elements of $\underline{a} \cap \underline{b}$ from the sets in the families $\underline{\beta}^*$ and $\underline{\alpha}^*$: in other words we can assume that $\underline{a} \cap \underline{b} = \emptyset$, and therefore $s = a = b$.

Therefore it is enough to prove the lemma for sets $\underline{a}, \underline{b}$ such that $\underline{a} \subseteq [n-c]$, $\underline{b} \subseteq [n] \setminus [n-c]$. Therefore we only need to do it for s, $0 \leqslant s \leqslant c$. Also, if $s = c$, then any injection will suffice (and we know an injection exists because of the cardinalities of the families). Therefore it is enough to prove the lemma for s, $0 \leqslant s \leqslant c-1$

If $s = 0$, then it is obviously true.

If $s = 1$, then $\underline{a} = \{i\}$ and $\underline{b} = \{j\}$ for some i, $1 \leqslant i \leqslant n-c$, and j, $n-c < j \leqslant n$. For a

$\underline{c}_1 \in \underline{\alpha}^*$ we have $i \in \underline{c}_1$. Let

$$\chi(\underline{c}_1) = \begin{cases} \underline{c}_1, & \text{if } j \in \underline{c}_1, \\ (\underline{c}_1 \setminus \{i\}) \cup \{j\}, & \text{if } j \notin \underline{c}_1. \end{cases}$$

It is easy to see that $\chi(\underline{c}_1) \in \underline{\beta}^*$, and χ is an injection satisfying the required property.

So assume $s > 1$, and we know the lemma for all s', $0 \leqslant s' < s$. Let $i \in \underline{a}$, and let $j \in [n] \setminus ([n-c] \cup \underline{b})$ (we know such a j exists because $b = s \leqslant c-1$). Define $\underline{a}' = (\underline{a} \setminus \{i\}) \cup \{j\}$. Let $\underline{\gamma}^* = \{\underline{c} : \underline{a}' \subset \underline{c}, |\underline{c}| = n-c+1\}$. By the induction hypothesis we have suitable injections from $\underline{\alpha}^*$ to $\underline{\gamma}^*$ and from $\underline{\gamma}^*$ to $\underline{\beta}^*$, and the composition of these injections will give χ. □

9. Maximum size of intersecting k-chains in $\mathfrak{b}^c_{n,k}$ and $\mathfrak{B}^c_{n,k}$

Lemma 9.1. *Let $c \in \{0,1\}$ and n,k be positive integers such that $k \leqslant n+1-2c$. There is a bijection ψ^* from $\mathfrak{t}^c_{n,k}[n-c]$ to $\mathfrak{t}^c_{n,k}[c]$ such that, for a $\underline{\gamma} = (\underline{l}^1 \subset \underline{l}^2 \subset \cdots \subset \underline{l}^k)$ such that $\underline{\gamma} \in \mathfrak{t}^c_{n,k}[n-c]$, we have that, if $\phi^*(\underline{\gamma}) = \underline{\delta}$ for some $\underline{\delta} = (\underline{m}^1 \subset \underline{m}^2 \subset \cdots \subset \underline{m}^k)$, then $\mathfrak{g}(\underline{\gamma}) = \mathfrak{g}(\underline{\delta})$, and for every i, $1 \leqslant i \leqslant k$ we have $|\underline{l}^i| \geqslant |\underline{m}^i|$.*

Proof. Let $\underline{\gamma} \in \mathfrak{t}^c_{n,k}[n-c]$, such that $\underline{\gamma} = (\underline{l}^1 \subset \cdots \subset \underline{l}^k)$. For $c = 0$ let

$$\psi(\underline{\gamma}) = \begin{cases} \underline{\gamma}, & \text{if } l^1 = \emptyset, \\ (\emptyset \subset \underline{l}^1 \subset \cdots \subset \underline{l}^{k-1}), & \text{if } \underline{l}^1 \neq \emptyset. \end{cases}$$

Since $\underline{l}^{k-1} \neq [n]$, it is easy to see that ψ is an injection from $\mathfrak{t}_{n,k}[c]$ to $\mathfrak{t}_{n,k}[n]$. To show that the required properties hold, it is enough to show that, whenever $\underline{\gamma} \neq \psi(\underline{\gamma})$, we also have $\mathfrak{g}(\underline{\gamma}) = \mathfrak{g}(\psi(\underline{\gamma}))$, since the rest is trivial. But in the case $\underline{\gamma} \neq \psi(\underline{\gamma})$ (using the fact that for any $r \in [n] = \underline{l}^k$ we have $[r] = [n] \cap [r]$, and the notation $\underline{l}^{-1} = \underline{l}^0 = \emptyset$),

$$\mathfrak{g}(\underline{\gamma}) = \sum_{p=1}^{k} \sum_{r \in \underline{l}^p \setminus \underline{l}^{p-1}} \left|[r] \setminus \underline{l}^p\right| = \sum_{p=0}^{k-1} \sum_{r \in \underline{l}^p \setminus \underline{l}^{p-1}} \left|[r] \setminus \underline{l}^p\right| = \mathfrak{g}(\psi(\underline{\gamma})).$$

All we need to show is that ψ is surjective. Let $\underline{\delta} = (\emptyset \subset \underline{m}^2 \subset \underline{m}^3 \cdots \subset \underline{m}^k) \in \mathfrak{r}_{n,k}[0]$. Let

$$\underline{\gamma}_{\underline{\delta}} = \begin{cases} \underline{\delta}, & \text{if } \underline{m}^k = [n], \\ (\underline{m}^2 \subset \underline{m}^3 \subset \cdots \subset \underline{m}^k \subset [n]), & \text{if } \underline{m}^k \neq [n]. \end{cases}$$

It is easy to check that $\underline{\gamma}_{\underline{\delta}} \in \mathfrak{t}_{n,k}[n]$ and $\psi(\underline{\gamma}_{\underline{\delta}}) = \underline{\delta}$.

For $c = 1$, define the sets $\underline{m}^p$ as follows:

$$\underline{m}^p = \begin{cases} [1], & \text{if } p = 1, \\ [1] \cup \{j+1 : j \in \underline{l}^{p-1}\}, & \text{if } 1 < p \leqslant k. \end{cases}$$

Obviously $(\underline{m}^1 \subset \underline{m}^2 \subset \cdots \subset \underline{m}^p) \in \mathfrak{t}^1_{n,k}[1]$. We set $\psi(\underline{\gamma}) = (\underline{m}^1 \subset \cdots \subset \underline{m}^k)$. It is easy to see that ψ is an injection, and for p, $1 < p \leqslant k$, we have $|\underline{m}^p| = |\underline{l}^{p-1}| + 1 \leqslant |\underline{l}^p|$. Obviously

$|\underline{m}^1| = 1 \leqslant |\underline{l}^p|$. Using the notation $\underline{m}^0 = \underline{l}^0 = \emptyset$ we have

$$\begin{aligned} \mathfrak{g}(\underline{\gamma}) &= \sum_{p=1}^{k} \sum_{r \in \underline{l}^p \setminus \underline{l}^{p-1}} |[r] \setminus \underline{l}^p| = \sum_{p=1}^{k-1} \sum_{r \in \underline{l}^p \setminus \underline{l}^{p-1}} |[r+1] \setminus \underline{m}^{p+1}| \\ &= \sum_{p=2}^{k} \sum_{r \in \underline{m}^p \setminus \underline{m}^{p-1}} |[r] \setminus \underline{m}^p| = \sum_{p=1}^{k} \sum_{r \in \underline{m}^p \setminus \underline{m}^{p-1}} |[r] \setminus \underline{m}^p| \\ &= \mathfrak{g}(\psi(\underline{\gamma})). \end{aligned}$$

All we need to show is that ψ is surjective. Let $\underline{\delta} = ([1] \subset \underline{m}^2 \subset \underline{m}^3 \cdots \subset \underline{m}^k) \in \mathfrak{r}^1_{n,k}[1]$. Let

$$\underline{l}^p = \begin{cases} \{j-1 \,:\, j \in \underline{m}^{p+1}\}, & \text{if } 1 \leqslant p < k, \\ [n-1], & \text{if } p = k. \end{cases}$$

Set $\underline{\gamma}_{\underline{\delta}} = (\underline{l}^1 \subset \cdots \subset \underline{l}^k)$. It is easy to check that $\underline{\gamma}_{\underline{\delta}} \in \mathfrak{t}^1_{n,k}[n-1]$ and $\psi(\underline{\gamma}_{\underline{\delta}}) = \underline{\delta}$. □

Corollary 9.2. *For $c \in \{0,1\}$ we have that $|\mathfrak{T}^c_{n,k}[c]| = |\mathfrak{T}^c_{n,k}[n-c]|$.*

Proof. By Lemma 6.3 and Lemma 9.1 we have that $|\mathfrak{T}^c_{n,k}[n-c]| = \sum_{\underline{\gamma} \in \mathfrak{t}^c_{n,k}[n-c]} q^{\mathfrak{g}(\underline{\gamma})} = \sum_{\underline{\delta} \in \mathfrak{t}^c_{n,k}[c]} q^{\mathfrak{g}(\underline{\delta})} = |\mathfrak{T}^c_{n,k}[c]|$. □

Theorem 9.3. *Let $c \in \{0,1\}$ and n,k be positive integers such that $k \leqslant n-2c+1$. For every standard intersecting family $\mathfrak{r} \subseteq \mathfrak{b}^c_{n,k}$ there is an injection $\psi = \psi_{\mathfrak{r}}$ from $\mathfrak{r}$ to $\mathfrak{t}^c_{n,k}[c]$ such that, for every $\underline{\gamma} = (\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r}$ and $\underline{\delta} = (\underline{m}^1 \subset \ldots \subset \underline{m}^k) \in \mathfrak{t}^c_{n,k}[c]$ such that $\psi(\underline{\gamma}) = \underline{\delta}$, the following hold.*

(1) $\mathfrak{g}(\underline{\gamma}) \leqslant \mathfrak{g}(\underline{\delta})$.
(2) *For every j, $1 \leqslant j \leqslant k$, we have* $|\underline{l}^j \cap [n-c]| \geqslant |\underline{m}^j \cap [n-c]|$.
(3) *For every j, $1 \leqslant j \leqslant k$, we have* $|\underline{l}^j \setminus [n-c]| \leqslant |\underline{m}^j \setminus [n-c]|$.
(4) $|\underline{l}^k| \geqslant |\underline{m}^k|$.

Proof. We are going to prove the theorem with induction on n,k. If $k=1$, then $|\mathfrak{r}| = 1$, and since $\mathfrak{r}$ is standard intersecting, there is an integer i such that $c \leqslant i \leqslant n-c$ and $\mathfrak{r} = \{([i])\}$. Also $\mathfrak{t}^c_{n,1}[c] = \{([c])\}$, and it is easy to check that the theorem holds.

So assume that $1 < k \leqslant n-2c+1$ and we know that the theorem holds for every k', n' such that $1 \leqslant k' \leqslant n'-2c+1$, $k' \leqslant k$, $n' \leqslant n$, and $n'+k' < n+k$. We define $\mathfrak{r}'$ as in Lemma 8.2.

If $\mathfrak{r} = \mathfrak{r}'$, then in particular $\mathfrak{r} \subseteq \mathfrak{t}^c_{n,k}[n-c]$, and the theorem is an immediate corollary of Lemma 9.1.

So assume $\mathfrak{r} \neq \mathfrak{r}'$. Let $\mathfrak{y} = \mathfrak{t}^c_{n,k}[c]$. By Lemma 8.3 it is enough to define a suitable injection from $\mathfrak{r}^c_{n,k}(i)$ to $\mathfrak{y}^c_{n,k}(i)$ for every $i \in \{0,1,\ldots,k\}$, since these sets partition $\mathfrak{r}$ and $\mathfrak{y}$. That is what we will do.

Case 1. $1 \leqslant i \leqslant k-1$.
Obviously $\widehat{\mathfrak{y}^c_{n,k}(i)} = \mathfrak{t}^c_{\widehat{n,k-1}}[c]$. Therefore by the induction hypothesis there is an injection $\psi^* : \widehat{\mathfrak{r}^c_{n,k}(i)} \to \widehat{\mathfrak{y}^c_{n,k}(i)}$ satisfying properties (1)–(4).

Let ψ_1 be the inverse of the bijection $\widehat{\ }: \mathfrak{y}^c_{n,k}(i) \to \widehat{\mathfrak{y}^c_{n,k}(i)}$ and ψ_2 be the bijection $\widehat{\ }: \mathfrak{r}^c_{n,k}(i) \to \widehat{\mathfrak{r}^c_{n,k}(i)}$. Let $\psi : \psi_1 \circ \psi^* \circ \psi_2$. Clearly ψ is an injection from $\mathfrak{r}^c_{n,k}(i)$ to $\mathfrak{y}^c_{n,k}(i)$. Let $\underline{\gamma} = (\underline{l}^1 \subset \cdots \subset \underline{l}^k) \in \mathfrak{r}^c_{n,k}(i)$ and $\underline{\delta} = (\underline{m}^1 \subset \cdots \subset \underline{m}^k) \in \mathfrak{y}^c_{n,k}(i)$ such that $\phi(\underline{\gamma}) = \underline{\delta}$. Since for every $j : 1 \leqslant j \leqslant k; j \neq i$ we have that

$$\begin{aligned} \left|\underline{l}^j \cap [n-c]\right| &= \left|\underline{l}^j \,\widehat{\cap\, [n-c]}\right| \geqslant \left|\underline{m}^j \,\widehat{\cap\, [n-c]}\right| = \left|\underline{m}^j \cap [n-c]\right|, \\ \left|\underline{l}^j \setminus [n-c]\right| &= \left|\underline{l}^j \setminus \widehat{[n-c]}\right| \leqslant \left|\underline{m}^j \setminus \widehat{[n-c]}\right| = \left|\underline{m}^j \setminus [n-c]\right|, \\ \left|\underline{l}^i \cap [n-c]\right| &= \left|\underline{l}^{i-1} \,\widehat{\cap\, [n-c]}\right| \geqslant \left|\underline{m}^{i-1} \,\widehat{\cap\, [n-c]}\right| = \left|\underline{m}^i \cap [n-c]\right|, \\ \text{and} \qquad \left|\underline{l}^i \setminus [n-c]\right| &= 1 = \left|\underline{m}^i \setminus [n-c]\right|, \end{aligned}$$

properties (2) and (3) are immediate. Property (4) follows from the fact that $\left|\underline{l}^k\right| = 1 + \left|\widehat{\underline{l}^k}\right| \geqslant 1 + \left|\widehat{\underline{m}^k}\right| = \left|\underline{m}^k\right|$.

By Lemma 8.1 we have that $\mathfrak{g}(\underline{\delta}) = \mathfrak{g}(\widehat{\underline{\delta}}) + (n - c - \left|\underline{m}^i \cap [n-c]\right|) + \left|\underline{m}^{i-1} \setminus [n-c]\right|$, and $\mathfrak{g}(\underline{\gamma}) = \mathfrak{g}(\widehat{\underline{\gamma}}) + (n - c - \left|\underline{l}^i \cap [n-c]\right|) + \left|\underline{l}^{i-1} \setminus [n-c]\right|$, therefore property (1) also holds.

Case 2. $i = 0$

By the induction hypothesis there is an injection $\psi' : \widehat{\mathfrak{r}^c_{n,k}(0)} \to \widehat{\mathfrak{y}^c_{n,k}(0)}$ which satisfies properties (1)–(4). We will lift it to a suitable injection ψ as follows. Let $\underline{\gamma} \in \mathfrak{r}^c_{n,k}(0)$, and $\widehat{\underline{\delta}} = \phi(\underline{\gamma})$. Assume $\underline{\gamma} = (\underline{l}^1 \subset \cdots \subset \underline{l}^k)$ and $\widehat{\underline{\delta}} = (\widehat{\underline{m}^1} \subset \cdots \subset \widehat{\underline{m}^k})$. We will define a $\underline{\delta}_{\underline{\gamma}} \in \mathfrak{y}^c_{n,k}(0)$ such that $\widehat{\underline{\delta}_{\underline{\gamma}}} = \widehat{\underline{\gamma}}$, and for every $\underline{\gamma}_1 \neq \underline{\gamma}_2$ such that $\widehat{\underline{\gamma}_1} = \widehat{\underline{\gamma}_2}$ we will have $\underline{\delta}_{\underline{\gamma}_1} \neq \underline{\delta}_{\underline{\gamma}_2}$. Setting $\psi(\underline{\gamma}) = \underline{\delta}_{\underline{\gamma}}$ will then give an injection from $\mathfrak{r}^c_{n,k}(0)$ to $\mathfrak{y}^c_{n,k}(0)$.

If $n - c \notin \underline{l}^k$, then let $\underline{\delta}_{\underline{\gamma}} = \widehat{\underline{\delta}}$, and properties (1)–(4) obviously hold.

If for some i, $1 \leqslant i \leqslant k$, we have that $n - c \in \underline{l}^i$ and $n - c \notin \underline{l}^{i-1}$, then let $\underline{\delta}_{\underline{\gamma}} = \left(\widehat{\underline{m}^1} \subset \cdots \subset \widehat{\underline{m}^{i-1}} \subset \left(\widehat{\underline{m}^i} \cup \{n-c\}\right) \subset \cdots \subset \left(\widehat{\underline{m}^k} \cup \{n-c\}\right)\right)$, and we can see that properties (1)–(4) hold in a similar way to Case 1.

It is easy to see that $\underline{\delta}_{\underline{\gamma}}$ is defined as promised, that is, ψ is an injection.

Case 3. $i = k$

If $c = 0$, then by Lemma 8.3 we have $\mathfrak{r}^c_{n,k}(k) = \emptyset$, and there is nothing to prove. So without loss of generality assume $c = 1$. Let $\mathfrak{r}^*$ and $\mathfrak{y}^*$ be the families we obtain by removing the last set from every chain of $\widehat{\mathfrak{r}^c_{n,k}(k)}$ and $\widehat{\mathfrak{y}^c_{n,k}(k)}$, respectively. Obviously $\mathfrak{r}^*$ and $\mathfrak{y}^*$ are standard intersecting families of $(k-1)$-chains, $\mathfrak{r}^* \subset \mathfrak{b}^c_{\widehat{n},k-1}$ and $\mathfrak{y}^* = \mathfrak{t}^c_{\widehat{n},k-1}[c]$. So by the induction hypothesis there is an injection $\psi^* : \mathfrak{r}^* \to \mathfrak{y}^*$ that satisfies properties (1)–(4). We want to lift it to an injection $\psi' : \widehat{\mathfrak{r}^c_{n,k}(k)} \to \widehat{\mathfrak{y}^c_{n,k}(k)}$, which also satisfies properties (1)–(4). The rest will follow from the fact that, for any set $\underline{x} \subseteq [n]$, if $n - c \notin \underline{x}$, then $\left|\underline{x} \cap [n-c]\right| = \left|\widehat{\underline{x}} \cap \widehat{[n-c]}\right|$ and $\left|\underline{x} \setminus [n-c]\right| = \left|\widehat{\underline{x}} \setminus \widehat{[n-c]}\right|$.

So consider a chain $(\underline{l}^1 \subset \cdots \subset \underline{l}^{k-1}) \in \mathfrak{r}^*$ and the chain $(\underline{m}^1 \subset \cdots \subset \underline{m}^{k-1}) \in \mathfrak{y}^*$ such that $\psi^*((\underline{l}^1 \subset \cdots \subset \underline{l}^{k-1})) = (\underline{m}^1 \subset \cdots \subset \underline{m}^{k-1})$. We know the following:

$$\begin{aligned} n - 1 - c \geqslant \left|\underline{l}^{k-1}\right| &\geqslant \left|\underline{m}^{k-1}\right|, \\ \left|\underline{l}^{k-1} \cap \widehat{[n-c]}\right| &\geqslant \left|\underline{m}^{k-1} \cap \widehat{[n-c]}\right|, \\ \left|\underline{l}^{k-1} \setminus \widehat{[n-c]}\right| &\geqslant \left|\underline{m}^{k-1} \setminus \widehat{[n-c]}\right|. \end{aligned}$$

If $\underline{\gamma} \in \mathfrak{r}^{c}_{n,k}(k)$ is the chain which resulted in the chain $(\underline{l}^1 \subset \cdots \subset \underline{l}^{k-1}) \in \mathfrak{r}^*$, then the last set in $\underline{\gamma}$ is a set of cardinality $n-c$ which contains $\underline{l}^{k-1}$; similarly the chains from $\mathfrak{y}^*$ which give $(\underline{m}^1 \subset \cdots \subset \underline{m}^{k-1}) \in \mathfrak{y}^*$ correspond to supersets of $\underline{m}^{k-1}$ with cardinality $n-c$, and moreover $n-c = \left|[\widehat{n}]\right| - c + 1$. It is easy to check that, in order to lift ψ^* to a suitable ψ, if it is enough to give an injection χ from $\{\underline{c} \subset [\widehat{n}] \ : \underline{l}^k \subseteq \underline{c},\ \left|\underline{c}\right| = n-c\}$ to $\{\underline{c} \subset [\widehat{n}] \ : \underline{m}^k \subseteq \underline{c},\ \left|\underline{c}\right| = n-c\}$, such that, if $\chi(\underline{c}_1) = \underline{c}_2$, then $\left|\underline{c}_1 \cap [\widehat{n-c}]\right| \geqslant \left|\underline{c}_2 \cap [\widehat{n-c}]\right|$ and $\left|\underline{c}_1 \setminus [\widehat{n-c}]\right| \leqslant \left|\underline{c}_2 \setminus [\widehat{n-c}]\right|$; by Lemma 8.4 such an injection exists. □

Erdős, Seress and Székely generalized the Erdős–Ko–Rado-type theorem for $c > 1$ for intersecting k-chains of sets. My proof does not give the generalization of this stronger theorem to intersecting k-chains of subspaces for $c > 1$, but an observant reader may realize that only the proper generalization of Lemma 9.1 is needed to obtain this stronger theorem.

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