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Subspaces of ωω That Are Paracompact in Some Forcing Extension

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SUBSPACES OF ω_ω THAT ARE PARACOMPACT IN SOME FORCING EXTENSION

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ABSTRACT. We discuss when a subspace of ω_ω is paracompact in some forcing extension.

1. Introduction

Engelking and Lutzer proved in [?] that for a linearly ordered space X, the following are equivalent:

(1) X is paracompact.

(2) X does not contain a closed copy of a stationary subset of an uncountable regular cardinal.

Motivated by this theorem, we are interested in the problem that for a set of ordinals X, find a property φ so that the following are equivalent:

(1^*) X is paracompact in some forcing extension where the cardinality of X is preserved.

(2^*) X has a property φ.

We study three cases:

Case 1. $X \subseteq \omega_1$. (Theorem **??**)

Case 2. $X \subseteq \omega_n$ for some $n \in \omega$. (Theorem **??**)

Case 3. $X \subseteq \omega_\omega$. (Theorem **??**)

Case 1 is a simple theorem; we were not able to find a property φ for Case 2 and Case 3, but found a sufficient condition for (2^*) to imply (1^*). (There is substantial difficulty finding such a condition φ for Case 2 and Case 3. See Remark **??**.) The main idea in this paper is to make a non–paracompact subspace of an ordinal in the ground model paracompact in forcing extension. The author obtained this idea from [?, Theorem 1.7].

Notations 1.1. For a regular uncountable cardinal κ, let

$$\mathcal{S}(\kappa) = \{S \subseteq \kappa : S \text{ is a stationary subset of } \kappa\}.$$

For each limit ordinal α, we fix a monotonically increasing continuous map

$$f_\alpha : cf(\alpha) \to \alpha$$

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cofinal in α. Note that for α with $cf(\alpha) > \omega$, $X \cap \alpha$ contains a copy of a stationary subset of $cf(\alpha)$ iff $f_\alpha^{-1}[X \cap \alpha]$ is a stationary subset of $cf(\alpha)$. For a set of ordinals X, we write $f_\alpha^{-1}[X]$ instead of $f_\alpha^{-1}[X \cap \alpha]$ even if X is not a subset of α.

For a set A and a cardinal κ, let $[A]^\kappa = \{B \subseteq A : |B| = \kappa\}$, and $[A]^{<\kappa} = \{B \subseteq A : |B| < \kappa\}$.

Fix a regular cardinal λ such that $\lambda \gg \aleph_\omega$ and let H_λ be the collection of all sets of cardinality hereditarily less than λ.

Using these notations, we can restate the theorem of Engelking and Lutzer mentioned earlier in the case where X is a set of ordinals:

Theorem 1.2. [**?**, Theorem 2.3] *Let θ be an ordinal and $X \subseteq \theta$. The following are equivalent:*

(1) *X is paracompact.*
(2) *For each $\alpha \in \theta + 1 \setminus X$ with $cf(\alpha) > \omega$, $f_\alpha^{-1}[X] \notin \mathcal{S}(cf(\alpha))$.*

In particular, for $X \subseteq \omega_1$, the following are equivalent:

(1') *X is paracompact.*
(2') *X is not a stationary subset of ω_1.*

The poset $CU(S)$.

Let us define a notion of forcing which we use throughout this paper. We say S is a *fat* stationary subset of an uncountable regular cardinal κ iff for every closed unbounded (club) subset C of κ, $C \cap S$ contains closed subsets of any order type less than κ [**?**, p.644]. Note that every stationary subset of ω_1 is fat [**?**]. For a fat stationary subset S of a regular uncountable cardinal κ, we define the partially ordered set

$$CU(S) = \{p \subseteq S : |p| < \kappa \text{ and } p \text{ is a closed subset of } \kappa\},$$

ordered by end–extension. Due to the result by Abraham and Shelah, we have the following (they proved a more general case than that in Theorem **??**; the case where $n = 1$ was proved by Baumgartner, Malitz, and Reinhardt [**?**]):

Theorem 1.3. [**?**, Theorem 1] *If S is a fat stationary subset of ω_n and $\aleph_{n-1}^{<\aleph_{n-1}} = \aleph_{n-1}$, then the following are true:*

(1) *Forcing with $CU(S)$ adds a club subset C of ω_n such that $C \subseteq S$;*
(2) *Forcing with $CU(S)$ does not add new subsets of size $< \aleph_n$ (so it preserves the cardinals $\leq \aleph_n$);*
(3) *If $\aleph_n^{<\aleph_n} = \aleph_n$, then forcing with $CU(S)$ preserves the cardinals $> \aleph_n$.*

2. Case where $X \subseteq \omega_1$

Let us consider the case where $X \subseteq \omega_1$ first. Here is a technical lemma.

Lemma 2.1. *Suppose that κ is an uncountable regular cardinal and forcing with $\mathbb{P}$ preserves the cofinality of κ. If, in $\mathbf{V}^{\mathbb{P}}$, a set A contains a club subset of κ, then, in $\mathbf{V}$, A is a stationary subset of κ.*

Proof. Assume on the contrary that, in $\mathbf{V}$, A is not a stationary subset of κ. Then $\kappa \setminus A$ contains a club subset, say C, of κ. In $\mathbf{V}^{\mathbb{P}}$, C and a club subset contained in A would be disjoint club subsets of an uncountable regular cardinal κ, which is a contradiction. □

Theorem 2.2. *For $X \subseteq \omega_1$, the following are equivalent:*

(1) *X is paracompact in some forcing extension in which ω_1 is preserved.*
(2) *X is a co-stationary subset of ω_1; that is, $\omega_1 \setminus X \in \mathcal{S}(\omega_1)$.*

Proof. (1) $\implies$ (2): Suppose X is paracompact in $\mathbf{V}^{\mathbb{P}}$ for some notion of forcing $\mathbb{P}$. By Theorem **??**, in $\mathbf{V}^{\mathbb{P}}$, $\omega_1 \setminus X$ contains a club subset of ω_1. By Lemma **??**, in $\mathbf{V}$, $\omega_1 \setminus X \in \mathcal{S}(\omega_1)$.

(2) $\implies$ (1): Suppose that $\omega_1 \setminus X \in \mathcal{S}(\omega_1)$. Let $\mathbb{P} = CU(\omega_1 \setminus X)$; then forcing with $\mathbb{P}$ preserves ω_1 and adds a club subset C of ω_1 contained in $\omega_1 \setminus X$, rendering X non–stationary. Therefore, X is paracompact in $\mathbf{V}^{\mathbb{P}}$ by Theorem **??**. □

3. Case where $X \subseteq \omega_n$

In this section, we let X be a subspace of ω_n and, assuming GCH, find a sufficient condition for X to be paracompact in some cardinal–preserving forcing extension.

Theorem 3.1. *Assume GCH. Let $X \subseteq \omega_n$ for some $n \geq 1$. Suppose that for each i with $1 \leq i \leq n$, there exists a fat stationary subset S_i of ω_i such that for each $\alpha \in \omega_n + 1 \setminus X$ with $cf(\alpha) = \omega_i$, $S_i \cap f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_i)$. Then X is paracompact in some cardinal–preserving forcing extension.*

Proof. For i with $1 \leq i \leq n$, we let

$$\mathbb{P}(i) = CU(S_n) \times CU(S_{n-1}) \times \cdots \times CU(S_i).$$

Claim 3.2. $\mathbb{P}(1)$ is cardinal-preserving.

Proof of Claim **??**. We will show by (downward) induction that for each i with $1 \leq i \leq n$,

(1) forcing with $\mathbb{P}(i)$ does not add new subsets of size $< \aleph_i$ (so it preserves the cardinals $\leq \aleph_i$), and
(2) forcing with $\mathbb{P}(i)$ preserves the cardinals $> \aleph_i$.

Since $\aleph_{n-1}^{<\aleph_{n-1}} = \aleph_{n-1}$ and $\aleph_n^{<\aleph_n} = \aleph_n$ (by GCH), forcing with $\mathbb{P}(n)(= CU(S_n))$ does not add new subsets of cardinality $< \aleph_n$ and preserves cardinals $> \aleph_n$ by Theorem **??**.

Assume that $\mathbb{P}(i+1)$ satisfies (1) and (2), not adding new subsets of size $< \aleph_{i+1}$ and preserving the cardinals $> \aleph_{i+1}$. We shall show that $\mathbb{P}(i)$ satisfies (1) and (2). We have $\mathbf{V}^{\mathbb{P}(i+1)} \models (\aleph_{i-1}^{<\aleph_{i-1}} = \aleph_{i-1}$ and $\aleph_i^{<\aleph_i} = \aleph_i)$. Therefore, $\mathbf{V}^{\mathbb{P}(i+1)} \models$ (forcing with $CU(S_i)$ does not add new subsets of size $< \aleph_i$ and preserves cardinals $> \aleph_i$). Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}(i+1)$–name for $CU(S_i)$ constructed in $\mathbf{V}^{\mathbb{P}(i+1)}$. Then $\mathbb{P}(i+1) * \dot{\mathbb{Q}}$ does not add new subsets of size

$< \aleph_i$ and preserves cardinals $> \aleph_i$. Since $CU(S_i)$ is a subset of the power set of ω_i and forcing with $\mathbb{P}(i+1)$ does not add new subsets of size $\leq \aleph_i$, we actually have that $(CU(S_i))^{\mathbf{V}^{\mathbb{P}(i+1)}} = (CU(S_i))^{\mathbf{V}}$. Therefore, $\mathbb{P}(i+1) * \dot{\mathbb{Q}}$ and $\mathbb{P}(i+1) \times CU(S_i)(= \mathbb{P}(i))$ produce the same generic extension. Thus, $\mathbb{P}(i)$ satisfies (1) and (2). $\square$ (Claim **??**)

To show that X is paracompact in $\mathbf{V}^{\mathbb{P}(1)}$, fix $\alpha \in \omega_n + 1 \setminus X$ such that $cf(\alpha) = \omega_k$ for some $k \leq n$. We need to show that $\mathbf{V}^{\mathbb{P}(1)} \models f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_k)$.

Working in $\mathbf{V}^{\mathbb{P}(k+1)}$ (in $\mathbf{V}$ if $k = n$), we have $\aleph_{k-1}^{<\aleph_{k-1}} = \aleph_{k-1}$ and so forcing with $CU(S_k)$ adds a club subset of ω_k through S_k. Since $\mathbb{P}(k) = \mathbb{P}(k+1) \times CU(S_k)$, we have $\mathbf{V}^{\mathbb{P}(k)} \models$ (S_k contains a club subset of ω_k). In $\mathbf{V}^{\mathbb{P}(1)}$, S_k remains a club subset of ω_k and it is still true that $S_k \cap f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_k)$ ("non–stationary" is preserved by any forcing), which implies that $f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_k)$. $\square$

Remark 3.3. Abraham and Shelah gave an example of forcing which adds a club subset of ω_2 through a non–fat stationary set [**?**, Theorem 5]. So S_n being a fat stationary subset of ω_n in Theorem **??** is not a necessary condition for S_n to contain a club subset of ω_n in forcing extension. However, Stanley showed in [**?**] that there is no satisfactory first–order characterization of those subsets of ω_2 that have club subsets in an outer model in which ω_1 and ω_2 are preserved.

4. Case where $X \subseteq \omega_\omega$

In this section we consider the case where $X \subseteq \omega_\omega$. To make X paracompact, we would need to force with $CU(S_n)$ for every $n \geq 1$, where $S_n \in \mathcal{S}(\omega_n)$. The following lemma assures us that we can do so. The point is that for this iteration to work, S_n's have to be lined up nicely so that the set (**??**) in Lemma **??** is stationary in $[H_\lambda]^{\aleph_k}$. The poset $\mathbb{P}_\omega$ defined in Lemma **??** is essentially the one defined by Stanley in [**?**, p. 372].

Lemma 4.1. *Assume GCH. Suppose that $\{S_n : n \geq 1\}$ is a sequence of sets such that:*

- *S_n is a fat stationary subset of ω_n for each $n \geq 1$;*
- *For $\alpha \in S_n$ such that $cf(\alpha) = \omega_k$ for some $k \geq 1$ and α is a limit point of S_n, $f_\alpha^{-1}[S_n]$ contains a club subset of ω_k;*
- *For each $k \in \omega$,*

$$\{x \in [H_\lambda]^{\aleph_k} : (\forall n > k) \sup(x \cap \omega_n) \in S_n\} \tag{4.1}$$

is a stationary subset of $[H_\lambda]^{\aleph_k}$.

Let $\mathbb{P}_\omega = \Pi_{n \geq 1} CU(S_n)$. Then

1. *$\mathbb{P}_\omega$ is cardinal–preserving, and*
2. *In $\mathbf{V}^{\mathbb{P}_\omega}$, S_n contains a club subset of ω_n for each $n \geq 1$.*

Proof. Let us show (1); (2) follows from the definition of $\mathbb{P}_\omega$. Let

$$\mathbb{P}_k = \Pi_{1 \leq n \leq k} CU(S_n)$$

and

$$\mathbb{P}^{k+1} = \Pi_{k+1\leq n<\omega} CU(S_n).$$

Claim 4.2. $\mathbb{P}^{k+1}$ is ω_k–distributive (the intersection of ω_k many dense open subsets of $\mathbb{P}^{k+1}$ is dense).

Assuming that the claim is true, let us finish the proof of the lemma. The claim implies that forcing with $\mathbb{P}^{k+1}$ does not add new subsets of size $\leq \aleph_k$, so $\mathbf{V}^{\mathbb{P}^{k+1}} \models$ "$\mathbb{P}_k = (\mathbb{P}_k)^{\mathbf{V}}$." Therefore, $\mathbb{P}^{k+1} \times \mathbb{P}_k$ and $\mathbb{P}^{k+1} * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}^{k+1}$–name for $\mathbb{P}_k$ constructed in $\mathbf{V}^{\mathbb{P}^{k+1}}$, produce the same extension. Forcing with $\mathbb{P}^{k+1}$ preserves the cardinals $\leq \aleph_{k+1}$ (by the claim), and in $\mathbf{V}^{\mathbb{P}^{k+1}}$ forcing with $\mathbb{P}_k$ preserves the cardinals $\geq \aleph_{k+1}$ because $|\mathbb{P}_k| \leq \aleph_k$. Therefore, forcing with $\mathbb{P}^{k+1} * \dot{\mathbb{Q}}$ preserves $\aleph_{k+1}$, which implies that forcing with $\mathbb{P}_\omega = \mathbb{P}^{k+1} \times \mathbb{P}_k$ preserves $\aleph_{k+1}$ for each $k \geq 0$. Thus, forcing with $\mathbb{P}_\omega$ preserves the cardinals $\leq \aleph_\omega$. Since $|\mathbb{P}_\omega| \leq \aleph_\omega$, forcing with $\mathbb{P}_\omega$ preserves the cardinals $> \aleph_\omega$ as well. Now, it remains to show the claim.

Proof of Claim **??**. Fix $p^* \in \mathbb{P}^{k+1}$ and a sequence $\vec{D} = \{D_i : i < \omega_k\}$ of dense open subsets of $\mathbb{P}^{k+1}$. We shall find $q \leq p^*$ such that $q \in \bigcap\{D_i : i < \omega_k\}$. Choose $M \in [H_\lambda]^{\aleph_k}$ such that

- $M \prec H_\lambda$;
- $\sup(M \cap \omega_n) \in S_n$ for each $n > k$;
  (this is possible by the fact that the set (**??**) is stationary)
- $[M]^{<\aleph_k} \subseteq M$;
- $\{p^*, \mathbb{P}^{k+1}, \vec{D}\} \subseteq M$;
- $\omega_k \subseteq M$.

Case 1. $k = 0$

We will construct a descending sequence $\{p_i : i < \omega\}$ such that

- $p^* \geq p_0 \geq p_1 \geq \cdots$;
- $p_i \in M \cap D_i$ for each $i < \omega$;
- $\sup[\bigcup_{i<\omega} p_i(n)] = \sup(M \cap \omega_n)$ for each $n \geq 1$.

Take $p_0 \in M$ such that $p_0 \in D_0$ and $p_0 \leq p^*$. Enumerate $M = \{x_i : i < \omega\}$. Working in M, we can find $p_i \in D_i$ such that $p_i \leq p_{i-1}$ and for each $n \geq 1$ $\max p_i(n) > x_i$ if x_i is an ordinal and $x_i \in \omega_n$. Define p_ω so that for each $n \geq 1$, $p_\omega(n) = \left[\bigcup_{i<\omega} p_i(n)\right] \cup \{\sup(M \cap \omega_n)\}$. Then $p_\omega \in \mathbb{P}^1$ and $p_\omega \in D_i$ for each $i < \omega$, showing that $\mathbb{P}^1$ is ω–distributive.

Case 2. $k > 0$.

Enumerate $M = \{x_i : i < \omega_k\}$, and construct a sequence $\{M_i \in [M]^{<\aleph_k} : i < \omega_k\}$ so that

- $M_i \prec M$ for all $i < \omega_k$;
- $\{p^*, \mathbb{P}^{k+1}, \vec{D}\} \subseteq M_i$;
- $\{x_i : i < j\} \subseteq M_j$;
- $\{M_i : i < j\} \subseteq M_j$ and $\{M_i : i \leq j\} \in M_{j+1}$;
- $M_j = \bigcup_{i<j} M_i$ for each limit ordinal j.

For each $n \geq k+1$, we have obtained a continuous increasing sequence $\{\sup(M_i \cap \omega_n) : i < \omega_k\}$, which is cofinal in $\sup(M \cap \omega_n)$. Let $\alpha = \sup(M \cap \omega_n)$; then $cf(\alpha) = \omega_k$, and for each $n \geq k+1$, $f_\alpha^{-1}[S_n]$ contains a club subset of ω_k (by the hypothesis) so $f_\alpha[\omega_k] \cap S_n$ contains a club subset of α. Therefore, by taking a subsequence, we can obtain $\{M_i : i < \omega_k\}$ such that

- $\sup(M_i \cap \omega_n) \in S_n$ for each $n \geq k+1$ and $i < \omega_k$.

Choose a descending sequence $\{p_i \in \mathbb{P}^{k+1} : i < \omega_k\}$ such that

- $p^* \geq p_0 \geq p_1 \geq \cdots \geq p_\xi \geq \cdots$;
- $p_i \in M_{i+1}$;
- $p_{i+1} \in D_i$;
- $\max p_i(n) \geq \sup(M_i \cap \omega_n)$ for all $n \geq k+1$;

(The last item implies that if j is a limit ordinal, then $\sup[\bigcup_{i<j} p_i(n)] = \sup(M_j \cap \omega_n)$ for each $n \geq k+1$. So we can define p_j in M_{j+1} such that)

- if j is a limit ordinal, then $p_j(n) = [\bigcup_{i<j} p_i(n)] \cup \{\sup(M_j \cap \omega_n)\}$ for each $n \geq k+1$.

Finally, define p_{ω_k} so that

- $p_{\omega_k}(n) = \left[\bigcup_{i<\omega_k} p_i(n)\right] \cup \{\sup(M \cap \omega_n)\}$ for each $n \geq k+1$.

Then $p_{\omega_k} \leq p^*$ and $p_{\omega_k} \in D_i$ for all $i < \omega_k$, showing that $\mathbb{P}^{k+1}$ is ω_k–distributive. □

Here is the main result of this paper:

Theorem 4.3. *Assume GCH. Let $X \subseteq \omega_\omega$. Suppose that $\{S_n : n \geq 1\}$ is as in Lemma **??** and for each $\alpha \in \omega_\omega \setminus X$ with $cf(\alpha) = \omega_n$, $S_n \cap f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_n)$. Then X is paracompact in some cardinal–preserving forcing extension.*

Proof. Let $\mathbb{P}_\omega$ be as in Lemma **??**, and we work in $\mathbf{V}^{\mathbb{P}_\omega}$. To show that X is paracompact, fix $\alpha \in \omega_\omega \setminus X$ such that $cf(\alpha) = \omega_k$ for some $k \geq 1$. We still have $S_n \cap f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_k)$, and S_n contains a club subset of ω_n, which implies that $f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_k)$. Thus, X is paracompact by Theorem **??**. □

5. Example

In order to show that Theorem **??** is not an empty theorem, we will give an example of X and S_n's as in the theorem such that $X \setminus \omega_n$ is not paracompact for all $n \in \omega$. To do so, we look at two technical lemmas.

Lemma 5.1. *Suppose κ and μ are regular cardinals such that $\kappa < \mu$. Let S be a stationary subset of μ such that for every $\alpha \in S$, $cf(\alpha) = \kappa$; then $\mathcal{S} = \{x \in [\mu]^\kappa : \sup(x) \in S\}$ is a stationary subset of $[\mu]^\kappa$.*

Proof. Menas [**?**, Theorem 1.5] proved that for every club $\mathcal{C} \subseteq [\mu]^\kappa$, there exists a function $f : \mu \times \mu \to [\mu]^{\leq\kappa}$ such that $\mathcal{C}_f := \{x \in [\mu]^\kappa : \forall\langle\xi,\eta\rangle \in x \times x\ f(\langle\xi,\eta\rangle) \subseteq x\} \subseteq \mathcal{C}$. (Note that $\mathcal{C}_f$ is a club subset of $[\mu]^\kappa$.) It therefore suffices to show that $\mathcal{C}_f$ meets $\mathcal{S}$ for all such f. Assume on the contrary that for some f, $\mathcal{C}_f$ misses $\mathcal{S}$.

Claim 5.2. For each $\alpha \in S$, $\exists\langle\xi,\eta\rangle \in \alpha\times\alpha$ such that $f(\langle\xi,\eta\rangle) \nsubseteq \alpha$.

Proof of Claim. Looking for a contradiction, assume that for some $\alpha^* \in S$, $f(\langle\xi,\eta\rangle) \subseteq \alpha^*$ for all $\langle\xi,\eta\rangle \in \alpha^*\times\alpha^*$. Take an increasing sequence $\{\alpha_\xi : \xi < \kappa\}$ cofinal in α^*. We define A_i for $i < \kappa$. Let $A_0 = \{\alpha_0\}$. If $j = i+1$, then let $A_j = \bigcup\{f(\langle\xi,\eta\rangle) : \langle\xi,\eta\rangle \in A_i \times A_i\} \cup \{\alpha_j\}$. If j is a limit ordinal, then let $A_j = \left[\bigcup_{i<j} A_i\right] \cup \{\alpha_j\}$. Finally, let $A_\delta = \bigcup_{i<\delta} A_i$; then $A_\delta \in \mathcal{C}_f \cap \mathcal{S}$, which is a contradiction. □ (Claim **??**)

For each $\alpha \in S$, choose $\langle\xi_\alpha,\eta_\alpha\rangle \in \alpha\times\alpha$ such that $f(\langle\xi_\alpha,\eta_\alpha\rangle) \nsubseteq \alpha$. Since $\xi_\alpha < \alpha$ and $\eta_\alpha < \alpha$ for all $\alpha \in S$, by applying Fodor's Lemma twice we can find $\xi^* \in \mu$, $\eta^* \in \mu$ and a stationary set $S' \subseteq S$ such that for all $\alpha \in S'$, $f(\langle\xi^*,\eta^*\rangle) \nsubseteq \alpha$, which is a contradiction. □

Lemma 5.3. *Let S_n be a fat stationary subset of ω_n. Then for $k < n$,*

$$\{x \in [H_\lambda]^{\aleph_k} : \sup(x\cap\omega_n) \in S_n\}$$

is a stationary subset of $[H_\lambda]^{\aleph_k}$.

Proof. Since S_n is fat, $\{\alpha \in S_n : cf(\alpha) = \omega_k\}$ is a stationary subset of ω_n. By Lemma **??**, $\{x \in [\omega_n]^{\aleph_k} : \sup(x) \in S_n\}$ is a stationary subset of $[\omega_n]^{\aleph_k}$. Menas [**?**, Corollary 1.9] proved that for $A \subseteq B$ with $|A| > \kappa$, if $\mathcal{S}$ is a stationary subset of $[A]^\kappa$, then $\{x \in [B]^\kappa : x\cap A \in \mathcal{S}\}$ is a stationary subset of $[B]^\kappa$. Apply this result to the fact that $\omega_n \subseteq H_\lambda$. □

Example 5.4. We give an example of X and S_n's as in Theorem **??** such that $X\setminus\omega_n$ is not paracompact for all $n \in \omega$.

For each $n \geq 1$, fix a stationary subset $A_n \subseteq \omega_n \setminus (\omega_{n-1}+1)$ such that $cf(\alpha) = \omega_{n-1}$ for all $\alpha \in A_n$, and $\{\alpha \in \omega_n \setminus A_n : cf(\alpha) = \omega_{n-1}\}$ is also stationary. Let

$$X = \bigcup_{n\geq 1} A_n,$$

and for each $n \geq 1$, set

$$S_n = \omega_n \setminus A_n.$$

For every $n \geq 1$, $X\setminus\omega_{n-1}$ is not paracompact because $A_n \subseteq X\setminus\omega_{n-1}$ and A_n is a stationary subset of ω_n and $\sup(A_n) = \omega_n \notin X$.

Let $\alpha \in \omega_\omega\setminus X$ and suppose $cf(\alpha) = \omega_k$ for some $k \geq 1$. We will show that $f_\alpha^{-1}[X]\cap S_k \notin \mathcal{S}(\omega_k)$. We can find $n \geq k$ such that $\alpha \in \omega_{n+1}\setminus\omega_n$. If $\alpha = \omega_n$, then we may assume that f_α is the identity on ω_n and so $f_\alpha^{-1}[X]\cap S_n = X\cap S_n$, which is not in $\mathcal{S}(\omega_n)$ because $X\cap S_n \subseteq \omega_{n-1}$. Next, suppose $\alpha > \omega_n$. It suffices to show that $f_\alpha^{-1}[X\setminus\omega_n]$ has no limit point in itself, which implies that $f_\alpha^{-1}[X] \notin \mathcal{S}(\omega_k)$. Indeed, each point in $X\setminus\omega_n (= \bigcup_{i>n} A_i)$ has cofinality $\geq \omega_n$. On the other hand, $f_\alpha^{-1}[X\setminus\omega_n] \subseteq \omega_k$.

We show S_n's are as in Lemma **??**. It is easy to see that S_n is a fat stationary subset of ω_n [**?**, Lemma 1.2] and $f_\alpha^{-1}[S_n]$ contains a club subset of ω_k for each $\alpha \in S_n$ with $cf(\alpha) = \omega_k$. Now, fix $k \geq 0$; we shall show that

$$\mathcal{E}_1 = \{x \in [H_\lambda]^{\aleph_k} : (\forall n > k)\sup(x\cap\omega_n) \in S_n\}$$

is a stationary subset of $[H_\lambda]^{\aleph_k}$. Let

$$\begin{aligned}\mathcal{E}_2 \quad = \quad & \{x \in [H_\lambda]^{\aleph_k} : \sup(x \cap \omega_{k+1}) \in S_{k+1}\} \cap \\ & \{x \in [H_\lambda]^{\aleph_k} : (\forall n > k) \sup(x \cap \omega_n) \notin x\}.\end{aligned}$$

$\mathcal{E}_2$ is a stationary subset of $[H_\lambda]^{\aleph_k}$ because the first set on the right side is stationary by Lemma **??** and the second set is a club set. To observe that $\mathcal{E}_2 \subseteq \mathcal{E}_1$, let $x \in \mathcal{E}_2$ and $n > k+1$; then $cf(\sup(x \cap \omega_n)) \leq \omega_k < \omega_{n-1}$ so $\sup(x \cap \omega_n) \in S_n$ (because $\{\alpha \in \omega_n : cf(\alpha) < \omega_{n-1}\} \subseteq S_n$).

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