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Covering Properties and Cohen Forcing

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COVERING PROPERTIES AND COHEN FORCING

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Abstract. We will show that adding Cohen reals preserves the covering property that every open cover has a σ-$\mathcal{P}$ $\mathcal{Q}$ refinement and deduce that adding Cohen reals preserves covering properties such as paracompactness, subparacompactness and screenability.

1. Introduction

Let $\langle X, \tau \rangle$ be a topological space. In $\mathbf{V}^{\mathbb{P}}$ (forcing extension by $\mathbb{P}$), we define a new topology $\tau^{\mathbb{P}}$ on X:

$$\tau^{\mathbb{P}} = \{\bigcup \mathcal{S} : \mathcal{S} \subseteq \tau\}.$$

Note that in general $\tau^{\mathbb{P}} \supsetneq \tau$ because, in $\mathbf{V}^{\mathbb{P}}$, τ is no longer closed under arbitrary union. However, τ still serves as a base for $\tau^{\mathbb{P}}$.

Grunberg, Junqueira, and Tall proved in [?] that if $\langle X, \tau \rangle$ is paracompact, then so is $\langle X, \tau^{\mathbb{P}} \rangle$ when $\mathbb{P}$ is Cohen forcing; in other words, they showed that adding Cohen reals preserves paracompactness. Using their ideas, we show that adding Cohen reals preserves covering properties such as screenability and subparacompactness as well.

Throughout this paper, we let, for a regular cardinal κ,

$\mathbb{P}_\kappa = Fn(\kappa, 2)$ = the set of all finite partial functions from κ to 2.

Forcing with $Fn(\kappa, 2)$ adds κ–many Cohen reals (see e.g. [?] p.204). We assume that all spaces are Hausdorff.

2. Preservation of covering properties

We look at the idea of *approximating an open cover* $\dot{\mathcal{U}}$ *of* $\langle X, \tau^{\mathbb{P}_\kappa} \rangle$ *by open covers of* $\langle X, \tau \rangle$ *in the ground model*, which was used in [?] and [?]. The following combinatorial structure of $Fn(\kappa, 2)$, which is proved to exist by A. Dow, is crucial to our study.

Definition 2.1. ([?] Lemma 1.1) An *n–dowment* is a family $\mathcal{L}_n$ of finite antichains of $\mathbb{P}_\kappa$ such that

(i) For each maximal antichain A of $\mathbb{P}_\kappa$, there exists an $L \in \mathcal{L}_n$ such that $L \subseteq A$.

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(ii) For every $p \in \mathbb{P}_\kappa$ with $|dom(p)| \leq n$ and for every collection $\{L_i : i \leq n\} \subseteq \mathcal{L}_n$, there exist $q \in \mathbb{P}_\kappa$ and $r_i \in L_i$ for each $i \leq n$ such that $q \leq p$ and $q \leq r_i$ for all $i \leq n$.

We fix an n-dowment $\mathcal{L}_n$ for each $n \in \omega$ for the remainder of this paper.

Here is the idea used in [**?**] and [**?**]. Let $\dot{\mathcal{U}}$ be a $\mathbb{P}_\kappa$-name for an open cover of $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$. Since τ is a base for $\tau^{\mathbb{P}_\kappa}$, we have for each $x \in X$ that

$$1 \Vdash \text{“}(\exists W \in \tau)(\exists U \in \dot{\mathcal{U}})(x \in W \subseteq U).\text{”}$$

Therefore, for each $x \in X$, we can find a maximal antichain $A_x \subseteq \mathbb{P}_\kappa$ such that for each $p \in A_x$, there exists $W_p(x) \in \tau$ such that $x \in W_p(x)$ and $p \Vdash$ “$(\exists U \in \dot{\mathcal{U}})(W_p(x) \subseteq U)$.” For each $x \in X$ and $n \in \omega$, choose $A_{x,n} \in \mathcal{L}_n$ such that $A_{x,n} \subseteq A_x$. Set

$$V_n(x) = \bigcap \{W_p(x) : p \in A_{x,n}\},$$

and let

$$\mathcal{V}_n = \{V_n(x) : x \in X\}.$$

We use $\mathcal{V}_n$ frequently in the remainder of this paper assuming that it is constructed in the above way, and we say that $\mathcal{V}_n$ is *an open cover of* $\langle X, \tau\rangle$ *constructed from a* $\mathbb{P}_\kappa$*-name* $\dot{\mathcal{U}}$ *for an open cover of* $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ *and an n-dowment* $\mathcal{L}_n$. Let us prove two lemmas concerning $\mathcal{V}_n$.

Lemma 2.2. *For each $n \in \omega$, suppose that $\mathcal{V}_n = \{V_n(x) : x \in X\}$ is an open cover of $\langle X, \tau\rangle$ constructed from a $\mathbb{P}_\kappa$-name $\dot{\mathcal{U}}$ for an open cover $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ and an n-dowment $\mathcal{L}_n$. Then*

(1) For each $p \in A_{x,n}$, $p \Vdash$ “$(\exists U \in \dot{\mathcal{U}})(V_n(x) \subseteq U)$”;

(2) $(\forall x \in X)(\forall p \in \mathbb{P}_\kappa)(\exists n \in \omega)(\forall V \in \mathcal{V}_n$ with $x \in V)(\exists q \leq p)$
$[q \Vdash (\exists U \in \dot{\mathcal{U}})(V \subseteq U)]$.

Proof. For (1), note that for every $p \in A_{x,n}$, $p \Vdash$ “$(\exists U \in \dot{\mathcal{U}})(W_p(x) \subseteq U)$” and $V_n(x) \subseteq W_p(x)$.

To prove (2), fix $x \in X$ and $p \in \mathbb{P}_\kappa$. Take $n \geq |dom(p)|$ and let $x \in V_n(y)$ for some $y \in X$. By Definition **??**(ii), there are $q \in \mathbb{P}_\kappa$ and $r \in A_{y,n}$ such that $q \leq p$ and $q \leq r$. By (1) and the fact that $q \leq r$, we have $q \Vdash$ “$(\exists U \in \dot{\mathcal{U}})(V_n(y) \subseteq U)$.” □

Lemma 2.3. *Suppose $\mathcal{U} \in \mathbf{V}^{\mathbb{P}_\kappa}$ is an open cover of $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$. Then, in $\mathbf{V}$, there exists a collection $\{\mathcal{V}_n : n \in \omega\}$ of open covers of $\langle X, \tau\rangle$ such that whenever we have a family $\{\mathcal{H}_n : n \in \omega\} \in \mathbf{V}$ of covers of $\langle X, \tau\rangle$ such that $\mathcal{H}_n$* refines *$\mathcal{V}_n$ for each $n \in \omega$, we can find an open cover $\mathcal{W} \in \mathbf{V}^{\mathbb{P}_\kappa}$ of $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ such that $\mathcal{W} \subseteq \bigcup\{\mathcal{H}_n : n \in \omega\}$ and $\mathcal{W}$* refines *$\mathcal{U}$.*

Proof. Let $\dot{\mathcal{U}}$ be a $\mathbb{P}_\kappa$-name for $\mathcal{U}$ and let G be a $\mathbb{P}_\kappa$-generic filter (so $\dot{\mathcal{U}}_G = \mathcal{U}$). For each $n \in \omega$, suppose $\mathcal{V}_n = \{V_n(x) : x \in X\}$ is an open cover of $\langle X, \tau\rangle$

constructed from $\dot{\mathcal{U}}$ and an n-dowment $\mathcal{L}_n$. Let $\{\mathcal{H}_n : n \in \omega\}$ be as in the statement. For each $x \in X$, let

$$D_x = \{p \in \mathbb{P}_\kappa : (\exists n \in \omega)(\exists H \in \mathcal{H}_n)[x \in H \text{ and } p \Vdash (\exists U \in \dot{\mathcal{U}})(H \subseteq U)]\}.$$

To show that D_x is dense, fix $p \in \mathbb{P}_\kappa$. Find an $n \in \omega$ as in Lemma **??**(2), and pick $H_x \in \mathcal{H}_n$ so that $x \in H_x$. Since $\mathcal{H}_n$ refines $\mathcal{V}_n$, we can find $V \in \mathcal{V}_n$ such that $H_x \subseteq V$. By Lemma **??**(2), we can find $q \leq p$ such that $q \Vdash$ "$(\exists U \in \dot{\mathcal{U}})(V \subseteq U)$", showing that D_x is dense. For each $x \in X$, pick $p_x \in D_x \cap G$ and $H_x \in \bigcup\{\mathcal{H}_n : n \in \omega\}$ so that $x \in H_x$ and $p_x \Vdash$ "$(\exists U \in \dot{\mathcal{U}})(H_x \subseteq U)$", and set $\mathcal{W} = \{H_x : x \in X\}$. We have $\mathbf{V}[G] \models$ "$\mathcal{W}$ refines $\mathcal{U}$." □

Here is the main theorem.

Theorem 2.4. *Suppose that every open cover (in $\mathbf{V}$) of a space $\langle X, \tau\rangle$ has a σ-$\mathcal{P}$ $\mathcal{Q}$ refinement, where*

$\mathcal{P}$ can be locally finite, locally countable, point finite, point countable, discrete, disjoint or countable, and

$\mathcal{Q}$ can be open, closed or ________.

Then every open cover (in $\mathbf{V}^{\mathbb{P}_\kappa}$) of $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ has a σ-$\mathcal{P}$ $\mathcal{Q}$ refinement.

Proof. Let $\mathcal{U} \in \mathbf{V}^{\mathbb{P}_\kappa}$ be an open cover of $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ and let $\{\mathcal{V}_n : n \in \omega\}$ be as in Lemma **??**. For each $n \in \omega$, take a σ-$\mathcal{P}$ $\mathcal{Q}$ refinement $\mathcal{H}_n$ of $\mathcal{V}_n$; note that $\bigcup\{\mathcal{H}_n : n \in \omega\}$ is still a σ-$\mathcal{P}$ family. Take $\mathcal{W} \subseteq \bigcup\{\mathcal{H}_n : n \in \omega\}$ such that $\mathcal{W}$ refines $\mathcal{U}$ as in Lemma **??**. □

Some covering properties can be characterized as every open cover having a σ-$\mathcal{P}$ $\mathcal{Q}$ refinement as below (see e.g. [**?**]):

Proposition 2.5. *Every open cover of X has:*

(1) a σ-locally finite open refinement and X is regular iff it is paracompact,
(2) a σ-discrete closed refinement iff it is subparacompact,
(3) a σ-disjoint open refinement iff it is screenable,
(4) a σ-point finite open refinement iff it is σ-metacompact,
(5) a σ-locally countable open refinement iff it is σ-paraLindelöf,
(6) a σ-countable open refinement iff it is Lindelöf,
(7) a σ-point countable open refinement iff it is metaLindelöf.

Applying Theorem **??**, we can obtain:

Corollary 2.6. *Adding Cohen reals preserves the following covering properties:*

(1) paracompact (i.e., if $\langle X, \tau\rangle$ is paracompact, then so is $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$),
(2) subparacompact,
(3) screenable,
(4) σ-metacompact,
(5) σ-para-Lindelöf,
(6) Lindelöf,
(7) metaLindelöf.

Proof. For (1), any forcing preserves regularity. □

3. Preservation of the lack of covering properties

In [?] (Lemma 5.4), Dow, Tall and Weiss proved that "not metrizable", "not paracompact", "not developable" and "not subparacompact" are preserved by adding Cohen reals. And they note (p.121, [?]) that "*[w]e leave it as an exercise for the readers to examine the various weakenings of paracompactness and metrizability .... to see which of them fit into our general scheme.*" We carry out this exercise and prove that "not submetacompact" and "not G_δ-diagonal" are preserved through adding Cohen reals. In addition, we directly prove the preservation of "not metrizable" and "not paracompact" by adding Cohen reals (in [?], they proved the preservation of "not metrizable" and "not paracompact" via proving "not collectionwise normality" is preserved). We do so by generalizing Lemma 5.4 in [?].

Lemma 3.1. *(cf. Lemma 5.4 [?]) Suppose that $\mathcal{U} \in \mathbf{V}$ is an open cover of $\langle X, \tau\rangle$ and $\{\mathcal{U}_n : n \in \omega\} \in \mathbf{V}^{\mathbb{P}_\kappa}$ is a sequence of open covers of $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ such that $\mathcal{U}_n$ refines $\mathcal{U}$ for each $n \in \omega$. Then there exists a sequence $\{\mathcal{V}_n : n \in \omega\} \in \mathbf{V}$ of open covers of $\langle X, \tau\rangle$ such that:*

(a) For $S \subseteq U \in \mathcal{U}$ with $S \in \mathbf{V}$, if $st(S,\mathcal{U}_n) \subseteq U$ for some $n \in \omega$, then $st(S, \mathcal{V}_m) \subseteq U$ for some $m \in \omega$.

(b) For x and y in X with $x \neq y$, if $y \notin st(x,\mathcal{U}_n)$ for some $n \in \omega$, then $y \notin st(x, \mathcal{V}_m)$ for some $m \in \omega$.

(c) For any $x \in X$, if there exist $n \in \omega$ and finite $\mathcal{K} \subseteq \mathcal{U}$ such that if $x \in W \in \mathcal{U}_n$ then $W \subseteq U$ for some $U \in \mathcal{K}$, then there exists an $m \in \omega$ such that if $x \in V \in \mathcal{V}_m$ then $V \subseteq U$ for some $U \in \mathcal{K}$.

Proof. For a proof of (a), replace x by S in the proof of Lemma 5.4 in [?]. A proof of (b) is similar to that of (a). So let us prove (c). We may assume that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each $n \in \omega$. Let $\dot{\mathcal{U}}_n$ be a $\mathbb{P}_\kappa$-name for $\mathcal{U}_n$ for each $n \in \omega$, and fix a $\mathbb{P}_\kappa$-generic filter G and so $\mathcal{U}_n = (\dot{\mathcal{U}}_n)_G$ for each $n \in \omega$. Suppose that for each $n \in \omega$, $\mathcal{V}_n = \{V_n(x) : x \in X\}$ is an open cover of $\langle X, \tau\rangle$ constructed from $\dot{\mathcal{U}}_n$ and an n-dowment $\mathcal{L}_n$. Fix $x \in X$; then there exist $p \in G$, $n \in \omega$ and finite $\mathcal{K} \subseteq \mathcal{U}$ such that

$$p \Vdash \text{“}(x \in W \in \dot{\mathcal{U}}_n) \implies (\exists U \in \mathcal{K})(W \subseteq U).\text{”}$$

Pick an integer $m \geq \max\{n, |dom(p)|\}$. To show that $\mathcal{V}_m$ works, let $x \in V_m(y)$. By Definition **??**(ii), there are $q \in \mathbb{P}_\kappa$ and $r \in A_{x,m}$ such that $q \leq p$ and $q \leq r$. Since $q \leq r$ and by Lemma **??**(1), we have

$$q \Vdash \text{“}(\exists W \in \dot{\mathcal{U}}_m)(V_m(y) \subseteq W).\text{”}$$

Since $1 \Vdash$ "$\dot{\mathcal{U}}_m$ refines $\dot{\mathcal{U}}_n$" and $q \leq p$, we have

$$q \Vdash \text{“}(\exists W \in \dot{\mathcal{U}}_n)(\exists U \in \mathcal{K})(V_m(y) \subseteq W \subseteq U).\text{”}$$

We therefore have that $q \Vdash$ "$(\exists U \in \mathcal{K})(V_m(y) \subseteq U)$." By absoluteness, $V_m(y) \subseteq U$ for some $U \in \mathcal{K}$. □

Here we give characterizations of some topological properties (see e.g. [**?**] and [**?**]); we will show the lack of these properties are preserved through adding Cohen reals.

Proposition 3.2. *(1) $\langle X, \tau\rangle$ is* metrizable *iff there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of open covers such that for $x \in U \in \tau$, there exist $n \in \omega$ and an open nbhd W of x such that $st(W, \mathcal{V}_n) \subseteq U$.*

(2) $\langle X, \tau\rangle$ is developable *iff there exists a sequence of open covers $\{\mathcal{V}_n\}_{n\in\omega}$ such that for $x \in U \in \tau$, there is an $n \in \omega$ such that $st(x, \mathcal{V}_n) \subseteq U$.*

(3) $\langle X, \tau\rangle$ is paracompact *iff for every open cover $\mathcal{U}$ of X, there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of open covers such that for any $x \in X$, there exist $n \in \omega$ and an open nbhd W of x such that $st(W, \mathcal{V}_n) \subseteq U$ for some $U \in \mathcal{U}$.*

(4) $\langle X, \tau\rangle$ is subparacompact *iff for every open cover $\mathcal{U}$ of X, there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of open covers such that for any $x \in X$, there is an $n \in \omega$ such that $st(x, \mathcal{V}_n) \subseteq U$ for some $U \in \mathcal{U}$.*

(5) $\langle X, \tau\rangle$ is submetacompact *iff for every open cover $\mathcal{U}$ of X, there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of open covers such that for any $x \in X$, there exist $n \in \omega$ and finite $\mathcal{K} \subseteq \mathcal{U}$ such that if $x \in V \in \mathcal{V}_n$, then $V \subseteq U$ for some $U \in \mathcal{K}$.*

(6) $\langle X, \tau\rangle$ is G_δ-diagonal *iff there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of open covers such that for each x, $y \in X$ with $x \neq y$, there is an $n \in \omega$ such that $y \notin st(x, \mathcal{V}_n)$.*

Theorem 3.3. *Adding Cohen reals preserves the following properties:*

(1) not metrizable (i.e., if $\langle X, \tau\rangle$ is not metrizable, then $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ is not metrizable either),

(2) not developable,

(3) not paracompact,

(4) not subparacompact,

(5) not submetacompact,

(6) not G_δ-diagonal.

Proof. We apply Lemma **??** to each statement in Proposition **??**.

For (1), use Lemma **??**(a) with τ in place of $\mathcal{U}$ and W in place of S; then this shows that if $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ is metrizable, then so is $\langle X, \tau\rangle$.

For (2), use Lemma **??**(a) with τ in place of $\mathcal{U}$ and $\{x\}$ in place of S.

For (3), use Lemma **??**(a) with W in place of S.

For (4), use Lemma **??**(a) with $\{x\}$ in place of S.

Lemma **??**(c) shows that if $\langle X, \tau^{\mathbb{P}_\kappa}\rangle$ is submetacompact, then so is $\langle X, \tau\rangle$, proving (5).

For (6), use Lemma **??**(b) with τ in place of $\mathcal{U}$ □

Lastly, let us address a question.

Question. *Does adding Cohen reals preserve metacompactness?*

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