6-3-2009

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Publication Info
Preprint version *Topology Proceedings*, Volume 34, 2009, pages 161-166.

A SPACE TOPOLOGIZED BY FUNCTIONS FROM ω TO ω

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Abstract. Let $\mathcal{F} \subseteq {}^{\omega}\omega$, where ${}^{\omega}\omega$ is the set of all functions from ω to ω. Define a topological space $\langle X, \tau(\mathcal{F})\rangle$ such that $X = \{p^*\} \cup [\omega \times \omega]$, each point in $\omega \times \omega$ is isolated, and a neighborhood of p^* has the form $\{p^*\} \cup \{\langle n, m\rangle : n \geq k,\ m \geq f(n)\}$ for some $k \in \omega$ and $f \in \mathcal{F}$. We investigate $\langle X, \tau(\mathcal{F})\rangle$ where $\mathcal{F}$ is a dominating subfamily, an unbounded subfamily, or a bounded subfamily of ${}^{\omega}\omega$.

1. Definition of $\langle X, \tau(\mathcal{F})\rangle$

Let ${}^{\omega}\omega$ denote the set of all functions from ω to ω. For $f \in {}^{\omega}\omega$ and $g \in {}^{\omega}\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. We define a topological space $\langle X, \tau(\mathcal{F})\rangle$ in the following way: $\mathcal{F} \subseteq {}^{\omega}\omega$ has the property that for $f_1 \in \mathcal{F}$ and $f_2 \in \mathcal{F}$, there exists $f_3 \in \mathcal{F}$ such that $f_1 \leq^* f_3$ and $f_2 \leq^* f_3$. Let

$$X = \{p^*\} \cup [\omega \times \omega].$$

Each point in $\omega \times \omega$ is isolated, $p^* \notin \omega \times \omega$ and a neighborhood base at p^* is the collection of sets

$$\{\{p^*\} \cup f^{\uparrow}_{\geq n} : n \in \omega,\ f \in \mathcal{F}\},$$

where

$$f^{\uparrow}_{\geq n} = \{\langle i, j\rangle : i \geq n,\ j \geq f(i)\}.$$

The purpose of this paper is to investigate the topological spaces $\langle X, \tau(\mathcal{F})\rangle$ with various $\mathcal{F}$. Here are key definitions.

Definitions 1.1. $\mathcal{F}$ is a *dominating* subfamily of ${}^{\omega}\omega$ if for every $g \in {}^{\omega}\omega$, there exists $f \in \mathcal{F}$ such that $g \leq^* f$.

$\mathcal{F}$ is an *unbounded* subfamily of ${}^{\omega}\omega$ if for every $g \in {}^{\omega}\omega$, there exists $f \in \mathcal{F}$ such that $f \not\leq^* g$.

$\mathcal{F}$ is a *bounded* subfamily of ${}^{\omega}\omega$ if there exists $g \in {}^{\omega}\omega$ such that for every $f \in \mathcal{F}$, $f \leq^* g$.

Observe that for $\mathcal{F} \subseteq {}^{\omega}\omega$, exactly one of the following three cases occurs:

(1) $\mathcal{F}$ is dominating;

(2) $\mathcal{F}$ is not dominating, but it is unbounded;

2000 *Mathematics Subject Classification.* 54A10.

Key words and phrases. Dominating, unbounded, bounded.

(3) $\mathcal{F}$ is bounded.

We are interested in finding topological properties φ and ψ such that:

(1*) $\langle X, \tau(\mathcal{F})\rangle$ has the property φ if and only if $\mathcal{F}$ is dominating;

(2*) $\langle X, \tau(\mathcal{F})\rangle$ has neither φ nor ψ if and only if $\mathcal{F}$ is not dominating, but it is unbounded;

(3*) $\langle X, \tau(\mathcal{F})\rangle$ has the property ψ if and only if $\mathcal{F}$ is bounded.

Theorem **??**(3) gives φ, but for ψ, we only have a partial result (Corollary **??**). For this incompleteness, we will pose a question at the end (Question **??**).

2. Characterization of $\langle X, \tau(\mathcal{F})\rangle$

The main theorem (Theorem **??**) characterizes $\langle X, \tau(\mathcal{F})\rangle$ where $\mathcal{F}$ is a dominating subfamily of ${}^{\omega}\omega$. In order to prove the theorem, we need to first establish some lemmas.

Lemma 2.1. *Let S be a subspace of $\langle X, \tau(\mathcal{F})\rangle$ such that $p^* \notin S$; then the following conditions are equivalent:*

(1) $p^* \in \overline{S}$.

(2) $(\forall n \in \omega)(\forall f \in \mathcal{F})(S \cap f^{\uparrow}_{\geq n} \neq \emptyset)$.

Proof. Obvious. □

Notation 2.2. For each $i \in \omega$, set

$$C_i = \{i\} \times \omega.$$

We call C_i the ith column.

Lemma 2.3. *Let S be a subspace of $\langle X, \tau(\mathcal{F})\rangle$ such that $p^* \notin S$. If S meets only finitely many columns (i.e., $S \subseteq \bigcup\{C_i : i \leq k\}$ for some $k \in \omega$), then $p^* \notin \overline{S}$.*

Proof. If $S \subseteq \bigcup\{C_i : i \leq k\}$ for some $k \in \omega$, then $S \cap f^{\uparrow}_{\geq k+1} = \emptyset$ for every $f \in \mathcal{F}$. By Lemma **??**, $p^* \notin \overline{S}$. □

Let us state and prove the main theorem.

Theorem 2.4 (Main Theorem)**.** *For a space $\langle X, \tau(\mathcal{F})\rangle$, the following statements are equivalent:*

(1) *$\mathcal{F}$ is a dominating subfamily of ${}^{\omega}\omega$.*

(2) *If $C'_i \subseteq C_i$ for each $i \in \omega$ and $p^* \in \overline{\bigcup\{C'_i : i \in \omega\}}$, then the set $\{i \in \omega : |C'_i| = \aleph_0\}$ is infinite.*

(3) *There does NOT exist a collection of subspaces $\{S_n \subseteq X \backslash \{p^*\} : n \in \omega\}$ such that:*

(a) *for every infinite set $E \subseteq \omega$, $p^* \in \overline{\bigcup\{S_n : n \in E\}}$, and*

(b) *whenever $a_n \in S_n$ for each $n \in \omega$, $p^* \in \overline{\{a_n : n \in \omega\}}$.*

Proof. (1) $\Longrightarrow$ (2). Assume that the set $\{i \in \omega : |C_i'| = \aleph_0\}$ is finite. Take $n \in \omega$ so that for every $i \geq n$, $|C_i'| < \aleph_0$. Since $\mathcal{F}$ is dominating, we can find $f \in \mathcal{F}$ such that for each $i \geq n$, $f(i) > \max\{j \in \omega : \langle i, j\rangle \in C_i'\}$, and so for every $i \in \omega$, $C_i' \cap f^{\uparrow}_{\geq n} = \emptyset$. By Lemma **??**, $p^* \notin \overline{\bigcup\{C_i' : i \in \omega\}}$.

(2) $\Longrightarrow$ (3). Fix a collection of subspaces $\mathcal{S} = \{S_n \subseteq X \setminus \{p^*\} : n \in \omega\}$ that satisfies condition (a). Note that $S_n = \emptyset$ for at most finitely many $n \in \omega$, so without loss of generality, we may assume that $S_n \neq \emptyset$ for all $n \in \omega$. We will show that $\mathcal{S}$ does not satisfy condition (b).

Claim. There exists $a_n \in S_n$ for each $n \in \omega$ such that the set $\{a_n : n \in \omega\}$ meets each column in a finite set (i.e., for every $k \in \omega$, $|\{n \in \omega : a_n \in C_k\}| < \aleph_0$). Assuming the claim is true, let $C_i' = \{a_n : a_n \in C_i\}$. Since each C_i' is finite, it must be the case that $p^* \notin \overline{\bigcup\{C_i' : i \in \omega\}}$, which negates condition(b). Now it remains to prove the claim.

Proof of Claim. We pick $a_n \in S_n$ in the following way: Let

$$L = \{n \in \omega : (\exists k \in \omega)(S_n \subseteq \bigcup_{i \leq k} C_i)\}.$$

Case 1: $n \in L$.
Let $k = \max\{j \in \omega : S_n \cap C_j \neq \emptyset\}$. Choose $a_n \in S_n \cap C_k$.

Case 2: $n \in \omega - L$.
Note that in this case S_n meets infinitely many columns. Take any $k > n$ with $S_n \cap C_k \neq \emptyset$, and pick $a_n \in S_n \cap C_k$.

Fix $k \in \omega$, and we will show that the set $\{n \in \omega : a_n \in C_k\}$ is finite. In order to do so, we prove that both $\{n \in L : a_n \in C_k\}$ and $\{n \in \omega - L : a_n \in C_k\}$ are finite sets. First, observe that $\{n \in \omega - L : a_n \in C_k\} \subseteq k$. If the other set $\{n \in L : a_n \in C_k\}$ is infinite, then $L' := \{n \in L : S_n \subseteq \bigcup_{i \leq k} C_i\}$ would be infinite as well, but by Lemma **??** we have $p^* \notin \overline{\bigcup\{S_n : n \in L'\}}$, violating condition (a). Hence, the set $\{n \in L : a_n \in C_k\}$ must be finite.

(3) $\Longrightarrow$ (1). By contrapositive. Assume that $\mathcal{F}$ is not dominating; then we can find $g \in {}^{\omega}\omega$ such that for every $f \in \mathcal{F}$, $g \not\leq^* f$. Set

$$S_n = \{\langle n, m\rangle : m \geq g(n)\}.$$

We show that S_n satisfies (a) and (b). For (a), fix an infinite set $E \subseteq \omega$; then for every $f \in \mathcal{F}$ and $k \in \omega$, $f^{\uparrow}_{\geq k} \cap \bigcup\{S_n : n \in \omega\} \neq \emptyset$; by Lemma **??**, condition (a) holds. For (b), pick $a_n \in S_n$ for each $n \in \omega$, and define a function $h \in {}^{\omega}\omega$ such that $a_n = \langle n, h(n)\rangle$; then $g(n) \leq h(n)$ for all $n \in \omega$, and therfore $h \not\leq^* f$ for all $f \in \mathcal{F}$. The set $\{n \in \omega : h(n) > g(n)\}$ is infinite for each $f \in \mathcal{F}$, which implies that for all $f \in \mathcal{F}$ and $k \in \omega$, $f^{\uparrow}_{\geq k} \cap \{\langle n, h(n)\rangle : n \in \omega\} \neq \emptyset$. Thus, by Lemma **??**, $p^* \in \overline{\{\langle n, h(n)\rangle : n \in \omega\}}$. ☐

Corollary 2.5. *The following statements are equivalent.*

(1) *$\mathcal{F}$ is a dominating subfamily of ${}^{\omega}\omega$.*
(2) *$\langle X, \tau(\mathcal{F})\rangle$ is homeomorphic to $\langle X, \tau({}^{\omega}\omega)\rangle$.*

Proof. (1) $\Longrightarrow$ (2). The identity map $id : \langle X, \tau(\mathcal{F})\rangle \to \langle X, \tau({}^\omega\omega)\rangle$ is a homeomorphism.

(2) $\Longrightarrow$ (1). If $\langle X, \tau(\mathcal{F})\rangle$ and $\langle X, \tau({}^\omega\omega)\rangle$ are homeomorphic, then $\langle X, \tau(\mathcal{F})\rangle$ satisfies condition (3) in Theorem **??**. □

Now let us turn our attention to the case where $\mathcal{F}$ is a bounded subfamily of ${}^\omega\omega$. First, we extend the definition of $\leq^*$ to every infinite subset of ω.

Definition 2.6. For $f, g \in {}^\omega\omega$ and an infinite set $E \subseteq \omega$, we define $f{\restriction}E \leq^* g{\restriction}E$ if the set $\{n \in E : f(n) > g(n)\}$ is finite.

Proposition 2.7. *For a space $\langle X, \tau(\mathcal{F})\rangle$, the following statements are equivalent:*

1. *$\mathcal{F}$ is bounded on some infinite subset of ω (i.e.,* $(\exists$ infinite $E \subseteq \omega)(\exists g \in {}^\omega\omega)(\forall f \in \mathcal{F})(f{\restriction}E \leq^* g{\restriction}E)$*).*
2. *There is a sequence in $X \setminus \{p^*\}$ converging to p^*.*

Proof. (1) $\Longrightarrow$ (2). Let $E \subseteq \omega$ and $g \in {}^\omega\omega$ be as in the statement. Enumerate the set $E = \{n_i : i \in \omega\}$ in increasing order. Set $a_i = \langle n_i, g(n_i)\rangle$ for each $i \in \omega$. To show that the sequence $\langle a_i : i \in \omega\rangle$ converges to p^*, take an arbitrary basic neighborhood $U = \{p^*\} \cup f^{\uparrow}_{\geq k}$ of p^*. Pick $k' \geq k$ so that for all $i \geq k'$ with $i \in E$, $f(n_i) \leq g(n_i)$; then for all $i \geq k'$, $a_i \in U$.

(2) $\Longrightarrow$ (1). Suppose that $\langle a_n : n \in \omega\rangle$ is a sequence converging to p^* such that $a_n \neq p^*$ for all $n \in \omega$. Let $S = \{a_n : n \in \omega\}$. Since $p^* \notin S$ and $p^* \in \overline{S}$, S meets infinitely many columns by (the contrapositive of) Lemma **??**. We can therefore take a subsequence $\langle a_{n_i} : i \in \omega\rangle$ that hits each column at most once (i.e., for each $k \in \omega$, $|\{i \in \omega : a_{n_i} \in C_k\}| \leq 1$). Let $E = \{k \in \omega : (\exists i \in \omega)(a_{n_i} \in C_k)\}$, and define $g \in {}^\omega\omega$ such that if $k \in E$, then $\langle k, g(k)\rangle = a_{n_i}$ for some $i \in \omega$, and if $k \in \omega - E$, then $g(k) = 0$. We show that E and g are as required. Suppose for a contradiction that for some $f \in \mathcal{F}$, $f{\restriction}E \not\leq^* g{\restriction}E$; this means $f(k) > g(k)$ for infinitely many $k \in E$, so $\langle k, g(k)\rangle \notin \{p^*\} \cup f^{\uparrow}_{\geq 0}$ for infinitely many $k \in E$, which implies that $a_{n_i} \notin \{p^*\} \cup f^{\uparrow}_{\geq 0}$ for infinitely many $i \in \omega$, contradicting the fact that $\langle a_{n_i} : i \in \omega\rangle$ converges to p^*. □

We say that a function $f \in {}^\omega\omega$ is *nondecreasing* if whenever $n < m$, $f(n) \leq f(m)$. We use the following fact.

Fact 2.8 ([**?**], Fact 3.4)**.** *Suppose that $\mathcal{F}$ is an unbounded subfamily of ${}^\omega\omega$ such that for every $f \in \mathcal{F}$, there exists a nondecreasing function $f' \in \mathcal{F}$ with $f \leq^* f'$. Then $\mathcal{F}$ is unbounded on every infinite subset of ω (i.e., for every infinite set $E \subseteq \omega$ and every $g \in {}^\omega\omega$, there exists $f \in \mathcal{F}$ such that $f{\restriction}E \not\leq^* g{\restriction}E$).*

Corollary 2.9. *Let $\mathcal{F} \subseteq {}^\omega\omega$ be such that for every $f \in \mathcal{F}$, there exists a nondecreasing function $f' \in \mathcal{F}$ with $f \leq^* f'$. Then the following statements are equivalent for $\langle X, \tau(\mathcal{F})\rangle$.*

(1) *$\mathcal{F}$ is a bounded subfamily of* ${}^{\omega}\omega$.
(2) *There is a sequence in* $X \setminus \{p^*\}$ *converging to* p^*.

Proof. (1) $\Longrightarrow$ (2). By Proposition **??**.
(2) $\Longrightarrow$ (1). By Proposition **??**, $\mathcal{F}$ is bounded on some infinite set $E \subseteq \omega$. By (the contrapositive of) Fact **??**, $\mathcal{F}$ is a bounded subfamily of ${}^{\omega}\omega$. □

We do not know a topological characterization of $\langle X, \tau(\mathcal{F})\rangle$ when $\mathcal{F}$ does not have abundant nondecreasing functions. We therefore ask the following question:

Question 2.10. Are there families $\mathcal{F} \subseteq {}^{\omega}\omega$ and $\mathcal{F}' \subseteq {}^{\omega}\omega$ such that $\mathcal{F}$ is unbounded, $\mathcal{F}'$ is bounded, yet $\langle X, \tau(\mathcal{F})\rangle$ and $\langle X, \tau(\mathcal{F}')\rangle$ are homeomorphic? If the answer is no, then what is a topological property ψ such that: $\langle X, \tau(\mathcal{F})\rangle$ has the property ψ if and only if $\mathcal{F}$ is bounded?

Acknowledgment. The author would like to thank Dr. Alan Dow for his valuable advice that has resulted in this paper.

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