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A Sequel to "A Space Topologized by Functions from Omega to Omega"

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A SEQUEL TO "A SPACE TOPOLOGIZED BY FUNCTIONS FROM ω TO ω"

TETSUYA ISHIU AND AKIRA IWASA

Abstract. We consider a topological space $\langle X, \tau(\mathcal{F})\rangle$, where $X = \{p^*\} \cup [\omega \times \omega]$ and $\mathcal{F} \subseteq {}^\omega\omega$. Each point in $\omega \times \omega$ is isolated and a neighborhood of p^* has the form $\{p^*\} \cup \{\langle i, j\rangle : i \geq n, j \geq f(i)\}$ for some $n \in \omega$ and $f \in \mathcal{F}$. We show that there are subsets $\mathcal{F}$ and $\mathcal{G}$ of ${}^\omega\omega$ such that $\mathcal{F}$ is not bounded, $\mathcal{G}$ is bounded, yet $\langle X, \tau(\mathcal{F})\rangle$ and $\langle X, \tau(\mathcal{G})\rangle$ are homeomorphic. This answers a question of the second author [**?**].

1. Introduction

Let us define a topological space $\langle X, \tau(\mathcal{F})\rangle$. ${}^\omega\omega$ denotes the set of all functions from ω to ω, and for $f \in {}^\omega\omega$ and $g \in {}^\omega\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Let $X = \{p^*\} \cup [\omega \times \omega]$, where $p^* \notin \omega \times \omega$, and let $\mathcal{F}$ be a subset of ${}^\omega\omega$ such that for any $f_1 \in \mathcal{F}$ and $f_2 \in \mathcal{F}$, there exists $f_3 \in \mathcal{F}$ such that $f_1 \leq^* f_3$ and $f_2 \leq^* f_3$. Each point in $\omega \times \omega$ is isolated and a neighborhood of p^* has the form $\{p^*\} \cup f^{\uparrow}_{\geq n}$ for some $n \in \omega$ and $f \in \mathcal{F}$, where

$$f^{\uparrow}_{\geq n} = \{\langle i, j\rangle : i \geq n, j \geq f(i)\}.$$

Recall that $\mathcal{F} \subseteq {}^\omega\omega$ is said to be a *dominating* family if for every $g \in {}^\omega\omega$ there exists an $f \in \mathcal{F}$ such that $g \leq^* f$, and that $\mathcal{F} \subseteq {}^\omega\omega$ is said to be a *bounded* family if there exists a $g \in {}^\omega\omega$ such that for every $f \in \mathcal{F}$, $f \leq^* g$.

In [**?**], the second author gave a topological characterization of the space $\langle X, \tau(\mathcal{F})\rangle$ when $\mathcal{F}$ is a dominating family, and asked if there is a topological characterization of $\langle X, \tau(\mathcal{F})\rangle$ when $\mathcal{F}$ is a bounded family. The purpose of this note is to answer this question negatively by constructing two families $\mathcal{F}$ and $\mathcal{G}$ such that

(1) $\mathcal{F}$ is not bounded in ${}^\omega\omega$,

(2) $\mathcal{G}$ is bounded in ${}^\omega\omega$, and

(3) $\langle X, \tau(\mathcal{F})\rangle$ and $\langle X, \tau(\mathcal{G})\rangle$ are homeomorphic.

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2. Theorem

In this section, we construct two spaces described in Introduction and prove that they have the required properties.

Theorem 2.1. *There exist subsets $\mathcal{F}$ and $\mathcal{G}$ of ${}^{\omega}\omega$ such that $\mathcal{F}$ is not bounded, $\mathcal{G}$ is bounded, and $\langle X, \tau(\mathcal{F})\rangle$ and $\langle X, \tau(\mathcal{G})\rangle$ are homeomorphic.*

Proof. Let $E = \{2n : n < \omega\}$ and $O = \{2n+1 : n < \omega\}$. Let $\mathcal{F}$ be the set of all $f \in {}^{\omega}\omega$ such that $f(2n) = 0$ for every $n < \omega$. Clearly, $\mathcal{F}$ is a non-dominating but unbounded family.

Claim 1. $Y \subseteq E \times \omega$ is $\tau(\mathcal{F})$-closed if and only if there exists an $\bar{n} < \omega$ such that $Y \subseteq \bar{n} \times \omega$.

$\vdash$ Clearly if $Y \subseteq \bar{n} \times \omega$ for some $\bar{n} < \omega$, then Y is $\tau(\mathcal{F})$-closed.

Suppose $Y \subseteq E \times \omega$ and there is no $\bar{n} < \omega$ such that $Y \subseteq \bar{n} \times \omega$. Let $f \in \mathcal{F}$ and $n < \omega$. By assumption, there exists an $\langle n', m\rangle \in Y$ such that $n' \geq n$. Since $Y \subseteq E \times \omega$, we have $n' \in E$. By the definition of $\mathcal{F}$, $f(n') = 0$. So, $\langle n', m\rangle \in f^{\uparrow}_{\geq n}$. Thus, $p^* \in \mathrm{cl}_{\tau(\mathcal{F})}(Y)$. Therefore, Y is not $\tau(\mathcal{F})$-closed. $\dashv$ (Claim 1)

Let $\pi : (O \times \omega) \to \omega$ be a bijection. Define A to be the set of all $a \subseteq \omega$ such that $\pi^{\leftarrow} a$ is $\tau(\mathcal{F})$-closed. For each $a \subseteq \omega$, define $g_a \in {}^{\omega}\omega$ by

$$g_a(n) = \begin{cases} 0 & \text{if } n \notin a \\ 1 & \text{if } n \in a \end{cases}$$

Let $\mathcal{G} = \{g_a : a \in A\}$. Note that $\mathcal{G}$ is bounded. To show that $\mathcal{G}$ is directed, pick g_a and g_b from $\mathcal{G}$. $\pi^{\leftarrow} a$ and $\pi^{\leftarrow} b$ are $\tau(\mathcal{F})$-closed so $\pi^{\leftarrow} a \cup \pi^{\leftarrow} b$ is $\tau(\mathcal{F})$-closed as well. Since $\pi^{\leftarrow} a \cup \pi^{\leftarrow} b = \pi^{\leftarrow}(a \cup b)$, $a \cup b \in A$. Clearly, $g_a \leq^* g_{a\cup b}$ and $g_b \leq^* g_{a\cup b}$.

We shall show that $\langle X, \tau(\mathcal{F})\rangle$ and $\langle X, \tau(\mathcal{G})\rangle$ are homeomorphic. Define a function $\nu : X \to X$ by

$$\begin{aligned} \nu(\langle 2n, m\rangle) &= \langle n, m+1\rangle \\ \nu(\langle 2n+1, m\rangle) &= \langle \pi(2n+1, m), 0\rangle \\ \nu(p^*) &= p^* \end{aligned}$$

Claim 2. $\nu : \langle X, \tau(\mathcal{F})\rangle \to \langle X, \tau(\mathcal{G})\rangle$ is a homeomorphism.

$\vdash$

Subclaim 2.1. For every $Y \subseteq \omega \times \omega$, if Y is $\tau(\mathcal{F})$-closed, then $\nu^{\rightarrow} Y$ is $\tau(\mathcal{G})$-closed.

$\vdash$ Suppose that Y is $\tau(\mathcal{F})$-closed. Since $\omega \times \omega$ is $\tau(\mathcal{F})$-discrete and $Y \subseteq \omega \times \omega$, $Y \cap (E \times \omega)$ is $\tau(\mathcal{F})$-closed. By Claim **??**, there exists an $\bar{n} < \omega$ such that $Y \cap (E \times \omega) \subseteq (2\bar{n}) \times \omega$. Then, by the definition of ν, $\nu^{\rightarrow}(Y \cap (E \times \omega)) \subseteq \bar{n} \times [1, \omega)$. Since $\bar{n} \times [1, \omega)$ is clearly $\tau(\mathcal{G})$-closed and $\omega \times \omega$ is $\tau(\mathcal{G})$-discrete, $\nu^{\rightarrow}(Y \cap (E \times \omega))$ is $\tau(\mathcal{G})$-closed.

We shall show that $\nu^{\to}(Y \cap (O \times \omega))$ is also $\tau(\mathcal{G})$-closed. Let $a = \pi^{\to}(Y \cap (O \times \omega))$. Then, since $\pi^{\leftarrow}a = Y \cap (O \times \omega)$ is $\tau(\mathcal{F})$-closed, we have $a \in A$. Let $\langle 2n{+}1, m\rangle \in Y \cap (O \times \omega)$. Then, $\nu(\langle 2n{+}1, m\rangle) = \langle \pi(2n{+}1, m), 0\rangle$. Note that $\pi(2n+1, m) \in a$. So, $g_a(\pi(2n+1, m)) = 1 > 0$. Hence, $\nu^{\to}(Y \cap (O \times \omega)) \cap (g_a)^{\uparrow}_{\geq 0} = \emptyset$. Therefore, $\nu^{\to}(Y \cap (O \times \omega))$ is $\tau(\mathcal{G})$-closed.
$\dashv$ (Subclaim 2.1)

Subclaim 2.2. For every $Y \subseteq \omega \times \omega$, if $\nu^{\to}Y$ is $\tau(\mathcal{G})$-closed, then Y is $\tau(\mathcal{F})$-closed.

$\vdash$ Suppose that $\nu^{\to}Y$ is $\tau(\mathcal{G})$-closed. Since $\nu^{\to}Y \subseteq \omega \times \omega$ and $\omega \times \omega$ is $\tau(\mathcal{G})$-discrete, both $(\nu^{\to}Y) \cap (\omega \times [1, \omega))$ and $(\nu^{\to}Y) \cap (\omega \times \{0\})$ are $\tau(\mathcal{G})$-closed.

Since $(\nu^{\to}Y) \cap (\omega \times [1, \omega))$ is $\tau(\mathcal{G})$-closed and for every $g \in \mathcal{G}$ and $n < \omega$, $g(n) \leq 1$, by a similar argument as Claim **??**, there exists an $\bar{n} < \omega$ such that $(\nu^{\to}Y) \cap (\omega \times [1, \omega)) \subseteq \bar{n} \times [1, \omega)$. By the definition of ν, it follows that $Y \cap (E \times \omega) \subseteq (2\bar{n}) \times \omega$. Thus, $Y \cap (E \times \omega)$ is $\tau(\mathcal{F})$-closed.

Since $(\nu^{\to}Y) \cap (\omega \times \{0\})$ is $\tau(\mathcal{G})$-closed, there exist $a \in A$ and $\bar{n} < \omega$ such that $(\nu^{\to}Y) \cap (\omega \times \{0\}) \cap (g_a)^{\uparrow}_{\geq \bar{n}} = \emptyset$. Note that $(\nu^{\to}Y) \cap (\omega \times \{0\}) = \nu^{\to}(Y \cap (O \times \omega))$. Let Y' be the set of all $\langle 2n{+}1, m\rangle \in Y \cap (O \times \omega)$ such that $\pi(2n{+}1, m) \geq \bar{n}$. Since π is a bijection, there are only finitely many elements $\langle 2n{+}1, m\rangle \in Y \cap (O \times \omega)$ such that $\pi(2n{+}1, m) < \bar{n}$. Hence, $(Y \cap (O \times \omega)) \setminus Y'$ is finite. So, in order to show that $(Y \cap (O \times \omega))$ is $\tau(\mathcal{F})$-closed, it suffices to show that Y' is $\tau(\mathcal{F})$-closed. To this end, it suffices to show that $Y' \subseteq \pi^{\leftarrow}a$ since $\pi^{\leftarrow}a$ is $\tau(\mathcal{F})$-closed and $\omega \times \omega$ is $\tau(\mathcal{F})$-discrete. Let $\langle 2n+1, m\rangle \in Y'$. Then, $\nu(\langle 2n+1, m\rangle) = \langle \pi(2n+1, m), 0\rangle$. Since $\langle \pi(2n+1, m), 0\rangle \notin (g_a)^{\uparrow}_{\geq \bar{n}}$ and $\pi(2n+1, m) \geq \bar{n}$, we have $g_a(\pi(2n+1, m)) \geq 1$. It follows that $\pi(2n+1, m) \in a$. So, $\langle 2n+1, m\rangle \in \pi^{\leftarrow}a$. $\dashv$ (Subclaim 2.2)

By these two subclaims, for every $Y \subseteq \omega \times \omega$, Y is $\tau(\mathcal{F})$-closed if and only if $\nu^{\to}Y$ is $\tau(\mathcal{G})$-closed. Therefore, by taking complements, for every $Z \subseteq X$ with $p^* \in Z$, Z is $\tau(\mathcal{F})$-open if and only if $\nu^{\to}Z$ is $\tau(\mathcal{G})$-open. If $Z \subseteq \omega \times \omega$, then Z is $\tau(\mathcal{F})$-open and $\nu^{\to}Z$ is $\tau(\mathcal{G})$-open because each point in $\omega \times \omega$ is isolated in both topologies. This shows that ν is a homeomorphism.
$\dashv$ (Claim 2)
$\square$(Theorem 2.1)

References

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