

2014

# Preservation of a Convergence of a Sequence to a Set

Akira Iwasa

*University of South Carolina - Beaufort, iwasa@uscb.edu*

Masaru Kada

*Osaka Prefectural University*

Shizuo Kamo

*Osaka Prefectural University*

Follow this and additional works at: [https://scholarcommons.sc.edu/beaufort\\_math\\_compscience\\_facpub](https://scholarcommons.sc.edu/beaufort_math_compscience_facpub)



Part of the [Applied Mathematics Commons](#)

## Publication Info

Preprint version *Topology Proceedings*, Volume 44, 2014, pages 97-105.

© Topology Proceedings 2014, Auburn University

This Article is brought to you by the Department of Mathematics and Computational Science at Scholar Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of Scholar Commons. For more information, please contact [dillarda@mailbox.sc.edu](mailto:dillarda@mailbox.sc.edu).

# PRESERVATION OF CONVERGENCE OF A SEQUENCE TO A SET

AKIRA IWASA, MASARU KADA, AND SHIZUO KAMO

ABSTRACT. We say that a sequence of points converges to a set if every open set containing the set contains all but finitely many terms of the sequence. We investigate preservation of convergence of a sequence to a set in forcing extensions.

## 1. INTRODUCTION

Let  $\langle X, \tau \rangle$  be a topological space and  $\mathbb{P}$  a notion of forcing. Let  $\mathbf{V}^{\mathbb{P}}$  denote the forcing extension of  $\mathbf{V}$  by  $\mathbb{P}$ . In  $\mathbf{V}^{\mathbb{P}}$ , we define a topology  $\tau^{\mathbb{P}} = \{\bigcup \mathcal{S} : \mathcal{S} \subseteq \tau\}$  on  $X$ , that is,  $\tau^{\mathbb{P}}$  is the topology generated by  $\tau$  in  $\mathbf{V}^{\mathbb{P}}$ . We say that a topological property  $\varphi$  is *preserved* by  $\mathbb{P}$  if whenever  $\langle X, \tau \rangle$  satisfies  $\varphi$ ,  $\langle X, \tau^{\mathbb{P}} \rangle$  also satisfies  $\varphi$ . First, let us observe the following:

**Theorem 1.1.** *Convergence of a sequence (to a point) is preserved by any forcing.*

*Proof.* If a sequence  $\{x_n : n \in \omega\}$  in  $\langle X, \tau \rangle$  converges to  $y$ , then in  $\langle X, \tau^{\mathbb{P}} \rangle$  it still converges to  $y$  because in  $\mathbf{V}^{\mathbb{P}}$   $\tau$  serves as a base for  $\tau^{\mathbb{P}}$ .  $\square$

By the above theorem, there is nothing to investigate about preservation of convergence of a sequence. So let us generalize the concept of convergence.

**Definition 1.2.** We say that a sequence of points  $\{x_n : n \in \omega\}$  *converges to a set*  $A$  if  $x_n \notin A$  for all  $n \in \omega$  and for every open set  $U$  containing  $A$ , there exists  $k \in \omega$  such that for every  $n \geq k$ ,  $x_n \in U$ .

Let us illustrate the fact that convergence of a sequence to a set is not necessarily preserved by forcing.

**Example 1.3.** There exists a space  $X$  and a sequence  $\{x_n : n \in \mathbb{N}\}$  in  $X$  such that:

- (1) in  $\mathbf{V}$ , the sequence  $\{x_n : n \in \mathbb{N}\}$  converges to a set  $A$ , and
- (2) in  $\mathbf{V}^{\mathbb{P}}$  for some forcing  $\mathbb{P}$ ,  $\{x_n : n \in \mathbb{N}\}$  does not converge to  $A$ .

*Proof.* Let  $[0, 1]^{\mathbf{V}}$  be the unit interval in  $\mathbf{V}$  equipped with the usual topology and  $\{q_n : n \in \mathbb{N}\}$  enumerate the rationals in  $[0, 1]^{\mathbf{V}}$ . In the example,

- $X = [0, 1]^{\mathbf{V}} \times [0, 1]^{\mathbf{V}}$  equipped with the usual topology.
- $x_n = (q_n, \frac{1}{n})$  for each  $n \in \mathbb{N}$ .
- $A = [0, 1]^{\mathbf{V}} \times \{0\}$ .

---

2010 *Mathematics Subject Classification.* 54A20, 03E40.

*Key words and phrases.* sequence, convergence, preservation, forcing.

The second author was supported by Grant-in-Aid for Young Scientists (B) 21740080, MEXT.

- $\mathbb{P}$  is any forcing notion which adjoins a new real.

The sequence  $\{(q_n, \frac{1}{n}) : n \in \mathbb{N}\}$  converges to the set  $[0, 1]^{\mathbf{V}} \times \{0\}$ . We shall show that in  $\mathbf{V}^{\mathbb{P}}$ , the sequence  $\{(q_n, \frac{1}{n}) : n \in \mathbb{N}\}$  does not converge to the set  $[0, 1]^{\mathbf{V}} \times \{0\}$ . Let  $r$  be a new real adjoined by  $\mathbb{P}$  such that  $0 < r < 1$ . Take subsequences  $\{q_{n_i} : i \in \mathbb{N}\}$  and  $\{q_{m_i} : i \in \mathbb{N}\}$  of rationals converging to  $r$  such that

$$q_{n_1} < q_{n_2} < q_{n_3} < \cdots < r < \cdots < q_{m_3} < q_{m_2} < q_{m_1}.$$

Consider the following points:

$$(0, 1), (q_{n_1}, \frac{1}{2n_1}), (q_{n_2}, \frac{1}{2n_2}), \cdots (r, 0) \cdots, (q_{m_2}, \frac{1}{2m_2}), (q_{m_1}, \frac{1}{2m_1}), (1, 1)$$

These points form a V-shape with  $(r, 0)$  being the tip of the letter V, and the tip  $(r, 0)$  is not in  $[0, 1]^{\mathbf{V}} \times \{0\}$ . Using the V-shape, we can define an open set containing the set  $[0, 1]^{\mathbf{V}} \times \{0\}$  and missing points  $(q_{n_i}, \frac{1}{n_i})$  and  $(q_{m_i}, \frac{1}{m_i})$  for all  $i \in \mathbb{N}$ . This implies that in  $\mathbf{V}^{\mathbb{P}}$  the sequence  $\{(q_n, \frac{1}{n}) : n \in \mathbb{N}\}$  does not converge to the set  $[0, 1]^{\mathbf{V}} \times \{0\}$ .  $\square$

In this note, we investigate under what circumstances convergence of a sequence to a set is preserved by forcing. Let us give definitions.

**Definition 1.4.** We say that a sequence  $\{x_n : n \in \omega\}$  is of *discrete points* if for each  $k \in \omega$ ,  $x_k \notin \overline{\{x_n : n \in \omega \setminus \{k\}\}}$ . (In this paper, we only consider sequences of discrete points.)

We write  $\{x_n\}_n$  for  $\{x_n : n \in \omega\}$  and  $\{x_{n_i}\}_i$  for  $\{x_{n_i} : i \in \omega\}$ .

A Tychonoff space  $X$  is said to be *pseudocompact* if every continuous real-valued function on  $X$  is bounded.

A point  $p$  is an *accumulation point* of a set  $A$  if for every neighborhood  $U$  of  $p$ ,  $U \cap A$  is infinite.

A point  $p$  is a *cluster point* of a family  $\{A_n : n \in \omega\}$  if for every neighborhood  $U$  of  $p$ ,  $U \cap A_n \neq \emptyset$  for infinitely many  $n \in \omega$ .

A space  $X$  is said to be *perfect* if for every  $x \in X$ ,  $x \in \overline{X \setminus \{x\}}$ .

$Fn(\kappa, 2)$  is the set of all finite partial functions from a cardinal  $\kappa$  to 2. ([8] Forcing with  $Fn(\kappa, 2)$  adjoins  $\kappa$ -many Cohen reals.)

A space  $X$  is said to be *scattered* if for every subspace  $S \subseteq X$ , there exists an  $x \in S$  such that  $x \notin \overline{S \setminus \{x\}}$ .

For a space  $X$ ,  $X$  can be uniquely represented as  $X = P \cup S$ , where  $P$  is a perfect set,  $S$  is a scattered set and  $P \cap S = \emptyset$ ; we say  $P$  is *the perfect kernel* of  $X$  and  $S$  is *the scattered kernel* of  $X$  ([3] Problem 1.7.10).

In this paper, we mainly deal with Tychonoff spaces. The property that a space is Tychonoff is preserved by any forcing ([2] Lemma 22).

## 2. CONVERGENCE OF A SEQUENCE TO A SET

In this section we study the concept of convergence of a sequence to a set. A proof of the following proposition is routine.

**Proposition 2.1.** *Let  $\{x_n\}_n$  be a sequence of discrete points in a space  $X$  and let  $A$  be a subset of  $X$  such that  $\{x_n\}_n \cap A = \emptyset$ . The following are equivalent:*

- (1)  $\{x_n\}_n$  converges to  $A$ .
- (2) For every subsequence  $\{x_{n_i}\}_i$  of  $\{x_n\}_n$ ,  $\overline{\{x_{n_i}\}_i} \cap A \neq \emptyset$ .

We use the following characterization of pseudocompact spaces.

**Proposition 2.2.** ([4] pp.177-178; [3] Theorem 3.10.23) *For a Tychonoff space  $X$ , the following are equivalent:*

- (1)  $X$  is pseudocompact.
- (2) If  $\mathcal{U} = \{U_n : n \in \omega\}$  is a sequence of nonempty open subsets of  $X$  such that  $U_i \cap U_j = \emptyset$  whenever  $i \neq j$ , then  $\mathcal{U}$  has a cluster point in  $X$ .
- (3) For every decreasing sequence  $U_1 \supseteq U_2 \supseteq \dots$  of non-empty open subsets of  $X$ ,  $\bigcap_{n \in \omega} \overline{U_n} \neq \emptyset$ .

The following are equivalent conditions for a sequence to converge to a set.

**Proposition 2.3.** *Suppose that  $X$  is a Tychonoff space and that  $\{x_n\}_n$  is a sequence of discrete points in  $X$ . The following are equivalent:*

- (1)  $\{x_n\}_n$  converges to a set.
- (2)  $\{x_n\}_n$  converges to the set  $\overline{\{x_n\}_n} \setminus \{x_n\}_n$ .
- (3) For every subsequence  $\{x_{n_i}\}_i$  of  $\{x_n\}_n$ ,  $\{x_{n_i}\}_i$  has an accumulation point, that is,  $\overline{\{x_{n_i}\}_i} \setminus \{x_{n_i}\}_i \neq \emptyset$ .
- (4) The closure  $\overline{\{x_n\}_n}$  of  $\{x_n\}_n$  is pseudocompact.

*Proof.* (1)  $\implies$  (2). Suppose that a sequence  $\{x_n\}_n$  does not converge to  $\overline{\{x_n\}_n} \setminus \{x_n\}_n$ . Take an arbitrary set  $A \subseteq X$  such that  $A \cap \{x_n\}_n = \emptyset$ . We shall show that the sequence  $\{x_n\}_n$  does not converge to  $A$ . By Proposition 2.1, there exists a subsequence  $\{x_{n_i}\}_i$  such that  $\overline{\{x_{n_i}\}_i} \cap (\overline{\{x_n\}_n} \setminus \{x_n\}_n) = \emptyset$ . Since  $\overline{\{x_{n_i}\}_i} \subseteq \overline{\{x_n\}_n}$ , it must be the case that  $\overline{\{x_{n_i}\}_i} \subseteq \{x_n\}_n$ . Since  $\{x_n\}_n \cap A = \emptyset$ , we have  $\overline{\{x_{n_i}\}_i} \cap A = \emptyset$ . By Proposition 2.1, the sequence  $\{x_n\}_n$  does not converge to  $A$ .

(2)  $\implies$  (3). Assume on the contrary that  $\overline{\{x_{n_i}\}_i} = \{x_{n_i}\}_i$ . Then  $\overline{\{x_{n_i}\}_i} \cap (\overline{\{x_n\}_n} \setminus \{x_n\}_n) = \emptyset$ , which implies that  $\{x_n\}_n$  does not converge to  $\overline{\{x_n\}_n} \setminus \{x_n\}_n$  by Proposition 2.1.

(3)  $\implies$  (4). Suppose  $Y := \overline{\{x_n\}_n}$  is not pseudocompact. According to Proposition 2.2, there exists in  $Y$  a family of nonempty open sets  $\mathcal{U} = \{U_i : i \in \omega\}$  such that  $U_i \cap U_j = \emptyset$  whenever  $i \neq j$  and  $\mathcal{U}$  has no cluster point in  $Y$ . For each  $i \in \omega$ , pick  $x_{n_i} \in U_i$ .  $\{x_{n_i}\}_i$  does not have an accumulation point in  $Y$ . Since  $Y$  is closed,  $\{x_{n_i}\}_i$  does not have an accumulation point in  $X$  either.

(4)  $\implies$  (1). It is clear that (2) implies (1), so it suffices to prove (4) implies (2). We assume on the contrary that  $\{x_n\}_n$  does not converge to  $\overline{\{x_n\}_n} \setminus \{x_n\}_n$ . By Proposition 2.1, there exists a subsequence  $\{x_{n_i}\}_i$  such that  $\overline{\{x_{n_i}\}_i} \cap (\overline{\{x_n\}_n} \setminus \{x_n\}_n) = \emptyset$ . This implies that  $\overline{\{x_{n_i}\}_i} \subseteq \{x_n\}_n$ . Since  $\{x_n\}_n$  is a sequence of discrete points, it follows that  $\overline{\{x_{n_i}\}_i} = \{x_{n_i}\}_i$ . Thus,  $\{\{x_{n_i}\}_i : i \in \omega\}$  is a family of open subsets of  $\overline{\{x_n\}_n}$  with no cluster point. By Proposition 2.2,  $\overline{\{x_n\}_n}$  is not pseudocompact.  $\square$

### 3. PRESERVATION OF CONVERGENCE TO A COMPACT SET

In this section, we investigate preservation of convergence of a sequence to a compact set. Let us look at a proposition.

**Proposition 3.1.** *Let  $\langle X, \tau \rangle$  be a compact space and  $\mathbb{P} = Fn(\kappa, 2)$  for some cardinal  $\kappa$ . The following are equivalent:*

- (1)  $\langle X, \tau^{\mathbb{P}} \rangle$  is compact.

- (2)  $\langle X, \tau^{\mathbb{P}} \rangle$  is countably compact.
- (3)  $\langle X, \tau^{\mathbb{P}} \rangle$  is pseudocompact.

*Proof.* Clearly (1)  $\implies$  (2)  $\implies$  (3). In order to show (3)  $\implies$  (1), we note that any forcing preserves regularity and that adjoining Cohen reals preserves Lindelöfness (cf. [5]). Therefore,  $\langle X, \tau^{\mathbb{P}} \rangle$  is a regular Lindelöf space and, in particular, it is normal ([3] Theorem 3.8.2). Every normal pseudocompact space is countably compact ([3] Theorem 3.10.21) and every countably compact Lindelöf space is compact.  $\square$

Here is a useful fact:

**Proposition 3.2.** ([6] Lemma 7; [1] Proposition 5.5) *For a compact Hausdorff space  $X$ , the following are equivalent:*

- (1) *The compactness of  $X$  is preserved by any forcing.*
- (2) *The compactness of  $X$  is preserved by adjoining a Cohen real.*
- (3)  *$X$  is scattered.*

Using Proposition 3.2, we obtain the theorem below.

**Theorem 3.3.** *Let  $X$  be a Tychonoff space. Suppose that a sequence  $\{x_n\}_n$  of discrete points in  $X$  converges to a compact set  $K$ . The following are equivalent:*

- (1) *In  $\mathbf{V}^{\mathbb{P}}$  with any forcing  $\mathbb{P}$ , the sequence  $\{x_n\}_n$  still converges to  $K$ .*
- (2) *In  $\mathbf{V}^{Fn(\omega, 2)}$ , the sequence  $\{x_n\}_n$  still converges to  $K$ .*
- (3) *The closure  $\overline{\{x_n\}_n}$  of  $\{x_n\}_n$  is scattered.*

*Proof.* (1)  $\implies$  (2) is obvious.

(2)  $\implies$  (3). Since  $\{x_n\}_n$  converges to a compact set, the closure  $\overline{\{x_n\}_n}$  is compact as well. Assume that  $\overline{\{x_n\}_n}$  is not scattered. Then  $\overline{\{x_n\}_n}$  is not compact in  $\mathbf{V}^{Fn(\omega, 2)}$  by Proposition 3.2. By Proposition 3.1,  $\overline{\{x_n\}_n}$  is not pseudocompact in  $\mathbf{V}^{Fn(\omega, 2)}$ . By Proposition 2.3, the sequence  $\{x_n\}_n$  does not converge to any set in  $\mathbf{V}^{Fn(\omega, 2)}$ .

(3)  $\implies$  (1). By Proposition 3.2,  $\overline{\{x_n\}_n}$  remains compact in  $\mathbf{V}^{\mathbb{P}}$ . Therefore, in  $\mathbf{V}^{\mathbb{P}}$ ,  $\{x_n\}_n$  converges to  $\overline{\{x_n\}_n} \setminus \{x_n\}_n$  by Proposition 2.3. It is not difficult to see that  $\overline{\{x_n\}_n} \setminus \{x_n\}_n \subseteq K$ . Thus, in  $\mathbf{V}^{\mathbb{P}}$ , the sequence  $\{x_n\}_n$  converges to  $K$ .  $\square$

The following example shows that the assumption of the compactness of  $K$  is necessary both in the implication (3) $\rightarrow$ (2) and (2) $\rightarrow$ (1) in Theorem 3.3. In Section 4, we remove the assumption of the compactness of  $K$  in the implication (1) $\rightarrow$ (3).

**Example 3.4.** ([9]) For an infinite almost disjoint family  $\mathcal{A}$  on  $\omega$ , we define a topological space  $\Psi(\mathcal{A})$  as follows. Let  $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$  as a set, each point from  $\omega$  is isolated, and a neighborhood base of  $A \in \mathcal{A}$  is the collection of sets of the form  $\{A\} \cup (A \setminus F)$  where  $F$  is a finite subset of  $A$ . Then  $\Psi(\mathcal{A})$  is a scattered space and we have  $\overline{\omega} = \Psi(\mathcal{A})$ .  $\Psi(\mathcal{A})$  is called the Mrówka space when  $\mathcal{A}$  is a maximal almost disjoint family. We consider  $\omega$  as a sequence of points in  $\Psi(\mathcal{A})$ . Note that  $\mathcal{A}$  is an infinite closed discrete subspace of  $\Psi(\mathcal{A})$  and so it is not compact.

**Claim 3.5.** The sequence  $\omega$  converges to  $\mathcal{A}$  in  $\Psi(\mathcal{A})$  iff  $\mathcal{A}$  is a maximal almost disjoint family.

*Proof.* Suppose that  $\mathcal{A}$  is not maximal and let  $X$  be an infinite subset of  $\omega$  which is almost disjoint from every  $A \in \mathcal{A}$ . Let  $U = (\omega \setminus X) \cup \mathcal{A}$ . Then  $U$  is an open set which contains  $\mathcal{A}$  and misses infinitely many points from  $\omega$ . On the other hand,

suppose that  $\omega$  does not converge to  $\mathcal{A}$ . Then there is an infinite subset  $X$  of  $\omega$  which has no accumulation point in  $\mathcal{A}$ . By the definition of a neighborhood of  $A \in \mathcal{A}$ , this means that  $X \cap A$  is finite for every  $A \in \mathcal{A}$ .  $\square$

Now we can easily observe the following fact. In  $\mathbf{V}$ , take a maximal almost disjoint family  $\mathcal{A}$  on  $\omega$ . Then  $\omega$  converges to  $\mathcal{A}$  in  $\Psi(\mathcal{A})$ . For a forcing notion  $\mathbb{P}$ , if  $\mathcal{A}$  remains maximal in  $\mathbf{V}^{\mathbb{P}}$ , then  $\omega$  still converges to  $\mathcal{A}$  in  $\mathbf{V}^{\mathbb{P}}$ , and otherwise  $\omega$  does not converge to  $\mathcal{A}$  in  $\mathbf{V}^{\mathbb{P}}$ .

In order to show that the assumption of the compactness of  $K$  is necessary in the implication (3) $\rightarrow$ (2) in Theorem 3.3, take a maximal almost disjoint family  $\mathcal{A}$  in  $\mathbf{V}$  obtained by extending the set of all branches through  $2^{<\omega}$  (identified with  $\omega$  through a bijection). According to [8] (VIII Exercise A14),  $\mathcal{A}$  is no longer maximal in  $\mathbf{V}^{\mathbb{P}}$ , where  $\mathbb{P}$  is any forcing notion which adjoins a new real. Therefore, the convergence of  $\omega$  to  $\mathcal{A}$  in  $\Psi(\mathcal{A})$  is destroyed by adjoining any new real.

To see that (2) $\rightarrow$ (1) in Theorem 3.3 does not hold without assuming  $K$  is compact, we note that if  $\mathbf{V}$  satisfies the continuum hypothesis, then in  $\mathbf{V}$  there is a maximal almost disjoint family  $\mathcal{A}$  on  $\omega$  which is still maximal in  $\mathbf{V}^{\text{Fn}(\omega, 2)}$  ([8] VIII Theorem 2.3). For such an  $\mathcal{A}$ , the convergence of  $\omega$  to  $\mathcal{A}$  in  $\Psi(\mathcal{A})$  is preserved by  $\text{Fn}(\omega, 2)$  but destroyed by some forcing (Maximality of any maximal almost disjoint family on  $\omega$  can be destroyed by forcing: Use the ccc poset defined in [8] (II Definition 2.7), or just collapse the cardinality of the family to  $\omega$ ).

#### 4. DESTROYING CONVERGENCE TO A SET

In this section, we show the implication (1) $\rightarrow$ (3) in Theorem 3.3 holds without assuming that  $K$  is compact. In other words, we define a forcing notion which can destroy convergence of a sequence  $\{x_n\}_n$  to a set, just assuming that  $\overline{\{x_n\}_n}$  is not scattered. First let us look at a lemma, which says adjoining a real makes a perfect space non-pseudocompact.

**Lemma 4.1.** *Suppose that  $\langle X, \tau \rangle$  is a Tychonoff space and that forcing with  $\mathbb{P}$  adjoins a real. If  $\langle X, \tau \rangle$  is perfect, then  $\langle X, \tau^{\mathbb{P}} \rangle$  is not pseudocompact.*

*Proof.* If  $\langle X, \tau \rangle$  is not pseudocompact, then there exists a continuous unbounded real-valued function  $f$  on  $X$ .  $f$  remains continuous and unbounded in  $\mathbf{V}^{\mathbb{P}}$  so  $\langle X, \tau^{\mathbb{P}} \rangle$  is not pseudocompact.

Now we assume that  $\langle X, \tau \rangle$  is pseudocompact. We construct a Cantor scheme ([7] Definition 6.1, Theorem 6.2), which is a family of non-empty open sets  $\{U_s : s \in 2^{<\omega}\}$  such that:

- (1)  $U_{s \smallfrown 0} \cap U_{s \smallfrown 1} = \emptyset$  for  $s \in 2^{<\omega}$ ;
- (2)  $\overline{U_{s \smallfrown i}} \subseteq U_s$  for  $s \in 2^{<\omega}$  and  $i \in \{0, 1\}$ .

Let  $U_{\emptyset} = X$ . Given  $U_s$  for  $s \in 2^{<\omega}$ , pick  $x \in U_s$  and  $y \in U_s$  with  $x \neq y$ . (This is possible because there is no isolated point.) Using regularity, choose open sets  $U_{s \smallfrown 0}$  and  $U_{s \smallfrown 1}$  such that  $U_{s \smallfrown 0} \cap U_{s \smallfrown 1} = \emptyset$ ,  $x \in U_{s \smallfrown 0} \subseteq \overline{U_{s \smallfrown 0}} \subseteq U_s$  and  $y \in U_{s \smallfrown 1} \subseteq \overline{U_{s \smallfrown 1}} \subseteq U_s$ .

Since  $X$  is pseudocompact,  $\bigcap \{\overline{U_s} : s \subseteq r\} \neq \emptyset$  for each  $r \in 2^\omega$  by Proposition 2.2. Working in  $\mathbf{V}^{\mathbb{P}}$ , take a generic real  $r^* \in 2^\omega \setminus \mathbf{V}$ .

**Claim 4.2.**  $\bigcap \{\overline{U_s} : s \subseteq r^*\} = \emptyset$ .

This claim implies that  $\langle X, \tau^{\mathbb{P}} \rangle$  is not pseudocompact by Proposition 2.2 and completes the proof of the lemma.

*Proof of Claim 4.2.* Assume on the contrary that  $\bigcap\{\overline{U_s} : s \subseteq r^*\} \neq \emptyset$  and pick  $x \in \bigcap\{\overline{U_s} : s \subseteq r^*\}$ . It is not difficult to see that for every  $r \in 2^\omega \cap \mathbf{V}$ ,  $x \notin \bigcap\{\overline{U_s} : s \subseteq r\}$ . Observe that

$$\bigcup_{r \in 2^\omega \cap \mathbf{V}} \left[ \bigcap\{\overline{U_s} : s \subseteq r\} \right] = \bigcap_{n \in \omega} \left[ \bigcup\{\overline{U_s} : s \in 2^n\} \right].$$

Since  $x$  does not belong to the set on the left-hand side,  $x \notin \bigcup\{\overline{U_s} : s \in 2^n\}$  for some  $n \in \omega$ . This implies that  $x \notin \overline{U_{r \upharpoonright n}}$ , which contradicts the fact that  $x \in \overline{U_s}$  for all  $s \subseteq r^*$ .  $\square$

Here is a crucial lemma:

**Lemma 4.3.** *Let  $X$  be a Tychonoff space. Suppose that:*

- (1) *a sequence  $\{x_i\}_i$  of discrete points of  $X$  converges to a set,*
- (2) *the closure  $\overline{\{x_i\}_i}$  of  $\{x_i\}_i$  is not scattered, and*
- (3) *the perfect kernel of  $\{x_i\}_i$  is not pseudocompact.*

*Then there is a forcing notion  $\mathbb{Q}$ , satisfying the countable chain condition, such that in  $\mathbf{V}^{\mathbb{Q}}$  the sequence  $\{x_i\}_i$  no longer converges to the set.*

*Proof.* Let  $P$  be the perfect kernel of  $\overline{\{x_i\}_i}$ . Since  $P$  is non-pseudocompact, we find by Proposition 2.2 a family  $\{V_n : n < \omega\}$  of pairwise disjoint nonempty open subsets of  $P$  without cluster point in  $P$ . For each  $n$  pick any point  $d_n$  from  $V_n$ , and set  $D = \{d_n : n < \omega\}$ . Then  $D$  does not accumulate anywhere in  $P$ , and neither does it in  $X$  since  $P$  is closed in  $X$ . We can find, for each  $n$ , a neighborhood  $U_n$  of  $d_n$  in  $X$  so that  $U_n$ 's are pairwise disjoint. For each  $n$  let  $\mathcal{U}_n$  be a neighborhood base of  $d_n$  inside  $U_n$  (e.g.,  $\mathcal{U}_n = \{V : V \text{ open in } X, d_n \in V \subseteq U_n\}$ ). Also, for each  $n$ , find a subset  $B_n$  of  $U_n \cap \{x_i\}_i$  such that  $d_n \in \overline{B_n}$ .

We define a forcing notion  $\mathbb{Q}$  by the following. A condition  $p$  of  $\mathbb{Q}$  is of the form  $p = (s^p, W^p)$ , where

- (1)  $s^p \in \bigcup_{l < \omega} (\prod_{n < l} B_n)$ , and
- (2)  $W^p \in \prod_{n < \omega} \mathcal{U}_n$ .

For  $p = (s^p, W^p)$ ,  $q = (s^q, W^q)$  in  $\mathbb{Q}$ ,  $p \leq_{\mathbb{Q}} q$  if:

- (1)  $s^p \supseteq s^q$
- (2) for all  $n < \omega$ ,  $W^p(n) \subseteq W^q(n)$ , and
- (3) for all  $n \in \text{dom}(s^p) \setminus \text{dom}(s^q)$ ,  $s^p(n) \in W^q(n)$ .

We observe that for each  $s \in \bigcup_{l < \omega} (\prod_{n < l} B_n)$ ,  $\{p \in \mathbb{Q} : s^p = s\}$  is centered (every finite subset has a lower bound), and therefore the set  $\mathbb{Q}$  ordered by  $\leq_{\mathbb{Q}}$  is a  $\sigma$ -centered (and hence ccc) forcing poset.

Let  $G$  be any  $\mathbb{Q}$ -generic filter over  $\mathbf{V}$ , and in  $\mathbf{V}[G]$  let  $S = \bigcup\{s^p : p \in G\}$ . For each  $n < \omega$ ,  $\{p \in \mathbb{Q} : |s^p| \geq n\}$  is dense in  $\mathbb{Q}$  and so  $S$  is an infinite subsequence of  $\{x_i\}_i$ .

**Claim 4.4.**  *$S$  does not accumulate anywhere in  $X$ .*

By Proposition 2.1, this claim implies that the sequence  $\{x_i\}_i$  does not converge to any set and hence finishes the proof of the theorem.

*Proof of Claim 4.4.* Fix  $x \in X$ . We shall show that  $x$  is not an accumulation point of  $S$ . Working in  $\mathbf{V}$ , since  $D$  has no accumulation point in  $X$ , we can find a neighborhood  $V$  of  $x$  such that  $\overline{V}$  meets  $D$  in at most one point. For each  $n < \omega$

with  $d_n \notin \bar{V}$ , choose  $H_n \in \mathcal{U}_n$  so that  $H_n \cap V = \emptyset$ . If  $d_n \in \bar{V}$ , then pick any  $H_n \in \mathcal{U}_n$ . The set  $\{p \in \mathbb{Q} : (\forall n < \omega)(W^p(n) \subseteq H_n)\}$  is dense in  $\mathbb{Q}$  so we can find  $p \in G$  such that  $W^p(n) \subseteq H_n$  for all  $n < \omega$ . This implies that  $S(n) \in W^p(n)$  for all  $n < \omega$ , and hence  $S$  does not accumulate at  $x$ .  $\square$

Combining Lemma 4.1 and Lemma 4.3, we prove the main theorem in this section.

**Theorem 4.5.** *Let  $X$  be a Tychonoff space. Suppose that:*

- (1) *a sequence  $\{x_i\}_i$  of discrete points in  $X$  converges to a set, and*
- (2) *the closure  $\overline{\{x_n\}_i}$  of  $\{x_i\}_i$  is not scattered.*

*Then there is a forcing notion  $\mathbb{P}$ , satisfying the countable chain condition, such that in  $\mathbf{V}^{\mathbb{P}}$  the sequence  $\{x_i\}_i$  no longer converges to the set.*

*Proof.* Let  $P$  be the perfect kernel of  $\overline{\{x_i\}_i}$ . By assumption,  $P \neq \emptyset$ . Let  $\mathbb{P}$  be a forcing notion which satisfies the ccc and adjoins a real. By Lemma 4.1, in  $\mathbf{V}^{\mathbb{P}}$ ,  $P$  is not pseudocompact. Let  $\dot{Q}$  be a  $\mathbb{P}$ -name for the poset defined in Lemma 4.3. Then a two-step iteration  $\mathbb{P} * \dot{Q}$  satisfies the ccc and in  $\mathbf{V}^{\mathbb{P} * \dot{Q}}$  the sequence  $\{x_i\}_i$  does not converge to the set by Lemma 4.3.  $\square$

#### REFERENCES

- [1] L. Aurichi, *D-spaces, topological games, and selection principles*, Top. Proc. **36** (2010), pp. 107–122.
- [2] J.E. Baumgartner, F.D. Tall, *Reflecting Lindelöfness*, Top. Appl. **122** (2002), pp. 35–49.
- [3] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.
- [4] K.P. Hart, J. Nagata, J.E. Vaughan (eds), *Encyclopedia of General Topology*, Elsevier Science, Amsterdam, 2004.
- [5] A. Iwasa, *Covering properties and Cohen forcing*, Top. Proc. **31** (2007), pp. 1–7.
- [6] I. Juhász and W. Weiss, *Omitting the cardinality of the continuum in scattered spaces*, Top. Appl. **31** (1989), no. 1, pp. 19–27.
- [7] A.S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.
- [8] K. Kunen, *Set Theory: An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.
- [9] S. Mrówka, *On completely regular spaces*, Fund. Math., **41** (1954), pp. 105–106.

UNIVERSITY OF SOUTH CAROLINA BEAUFORT, ONE UNIVERSITY BOULEVARD, BLUFFTON, SC 29909, USA

*E-mail address:* iwasa@uscb.edu

GRADUATE SCHOOL OF SCIENCE, OSAKA PREFECTURE UNIVERSITY, 1-1 GAKUEN-CHO, NAKA-KU, SAKAI, OSAKA 599-8531 JAPAN

*E-mail address:* kada@mi.s.osakafu-u.sc.jp

OSAKA PREFECTURE UNIVERSITY, 1-1 GAKUEN-CHO, NAKA-KU, SAKAI, OSAKA 599-8531 JAPAN

*E-mail address:* kamo@mi.s.osakafu-u.sc.jp