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A Family of Simple Codimension Two Singularities with Infinite Cohen-Macaulay Representation Type

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Abstract

A celebrated theorem of Buchweitz, Greuel, Knörrer, and Schreyer is that the hypersurface singularities of finite representation type, i.e. the hypersurface singularities admitting only finitely many indecomposable maximal Cohen-Macaulay modules, are exactly the ADE singularities. The codimension 2 singularities that are the analogs of the ADE singularities have been classified by Frühbis-Krühger and Neumer, and it is natural to expect an analogous result holds for these singularities. In this paper, I will present a proof that, in contrast to hypersurfaces, Frühbis-Krühger and Neumer's singularities include a subset of singularities of infinite representation type.

Table of Contents

List of Figures

Chapter 1

Introduction and Notation

The main theorem in this work follows in a long line of papers classifying rings based on the Cohen-Macaulay representation type of the ring. In the first chapter, we will go over some of these results, set up notation for the rest of the paper, and introduce the family of singularities that this paper pertains to. First, we will review the main definitions of the paper.

1.1 Introduction

Recall that a finitely generated module, M, over a Cohen-Macaulay local ring, $(R, \mathfrak{m})$, is *maximal Cohen-Macaulay* if $\operatorname{depth} M = \dim R$. For an arbitrary Noetherian ring we say a module is maximal Cohen-Macaulay if the localization at every maximal ideal is a maximal Cohen-Macaulay module. This class of modules inherits many properties of the ring it is over, making the study of this class of modules a tractable way to study a ring. We say a local ring, $(R, \mathfrak{m})$, has *finite Cohen-Macaulay representation type* if there are only finitely many indecomposable maximal Cohen-Macaulay R-modules, up to isomorphism. If there are infinitely many indecomposable non-isomorphic maximal Cohen-Macaulay modules over a ring, we say the ring has *infinite Cohen-Macaulay representation type.*

As mentioned above, there are many papers classifying rings based on Cohen-Macaulay representation type, and we recall some of those results now. As with the standard texts on this subject ([16],[17]), we will go over these theorems by increasing order of dimension, and then go over the partial classifications in higher dimensions.

Most of these results are true over a more general field than $\mathbb{C}$. However, since the main theorem of this paper is over $\mathbb{C}$, we will state these results over $\mathbb{C}$ as well.

Theorem 1.1 ([16])**.** *A complete Artinian local ring,* $(R, \mathfrak{m}, \mathbb{C})$ *has finite Cohen-Macaulay representation type if and only if it isomorphic to a principle ideal ring, i.e.* $R \cong \mathbb{C}[\![x]\!]/(x^n)$*.*

In the dimension 0 case, maximal Cohen-Macaulay means nothing more than finitely generated. For this case we can list the set of maximal Cohen-Macaulay modules.

Example 1.2. For the ring $R = \mathbb{C}[\![x]\!]/(x^n)$, the set of indecomposable maximal Cohen-Macaulay R-modules is

$$\left\{R, R/(x), \dots R/(x^{n-1})\right\}.$$

The ADE singularities are significant to the main result of this work and to Cohen-Macaulay representation theory, in general. For that reason, we recall the definition now.

Definition 1.3. We say R is the *local ring of an ADE hypersurface singularity*, or sometimes R is an *ADE hypersurface singularity* if $R \cong \mathbb{C}[\![x, y, z_1, \dots, z_n]\!]/(f)$, where f is one of the following equations:

Type	f
A_k	$x^{k+1} + y^2 + z_1^2 + \dots + z_n^2$, $k \geq 1$
D_k	$x^{k-1} + xy^2 + z_1^2 + \dots + z_n^2$, $k \geq 4$
E_6	$x^3 + y^4 + z_1^2 + \dots + z_n^2$
E_7	$x^3 + xy^3 + z_1^2 + \dots + z_n^2$
E_8	$x^3 + y^5 + z_1^2 + \dots + z_n^2$

Moving to Cohen-Macaulay representation type in dimension one, things are much more interesting.

Recall in dimension one, a module is maximal Cohen-Macaulay if and only if it is torsion-free, providing a well understood property to look for in modules.

Theorem 1.4 ([11])**.** *A reduced complete dimension one local ring with residue field $\mathbb{C}$ has finite Cohen-Macaulay representation type if and only if it is a finite birational extension of an ADE hypersurface singularity.*

Thus, in the reduced complete dimension one case, a ring has finite Cohen-Macaulay representation type if and only if it is a birational extension of $\mathbb{C}[\![x, y]\!]/(f)$, where f is one of the following equations.

Type	f
A_k	$x^{k+1} + y^2,\ k \geq 1$
D_k	$x^{k-1} + xy^2,\ k \geq 4$
E_6	$x^3 + y^4$
E_7	$x^3 + xy^3$
E_8	$x^3 + y^5$

The birational extensions of the ADE curve singularities are classified in [9]. Therefore, we have a reduced complete dimension one local ring over $\mathbb{C}$ is finite Cohen-Macaulay representation type if and only if it is the local ring of a curve singularity of type $A_k, D_k, E_6, E_7, E_8, A_k \bigvee L, E_6(1), E_7(1)$, or $E_8(1)$. The complete sets of indecomposable maximal Cohen-Macaulay modules over the ADE curve singularities are listed in [17] and [16]. Furthermore, the rank one indecomposable maximal Cohen-Macaulay modules over the birational extensions are documented in [14]. We look at one such example below.

Example 1.5. Let $R = \mathbb{C}[\![x, y, z]\!]/(xy, xz, yz)$. Then, R is the local ring of a type $A_1 \bigvee L$ space curve singularity. In particular, R is a birational extension of the D_4 curve singularity. Hence, R has finite Cohen-Macaulay representation type. The set of indecomposable non-isomorphic maximal Cohen-Macaulay R-modules is

$$\{R,(x+y,x+z),R/(x),R/(y),R/(z),R/(x,y),R/(x,z),R/(y,z)\}\,.$$

For dimension two rings of finite Cohen-Macaulay representation type, we have the following result:

Theorem 1.6 ([1],[8])**.** *If a dimension two complete Cohen-Macaulay ring, R, over $\mathbb{C}$ has finite Cohen- Macaulay representation type, then R is isomorphic to the local ring of a quotient singularity, i.e. R is isomorphic to $\mathbb{C}[\![u,v]\!]^G$ for some finite group $G \subset \mathrm{GL}_2(\mathbb{C})$.*

Recall that in dimension two, being maximal Cohen-Macaulay is equivalent to begin reflexive. As in the dimension one case, this provides a well studied condition equivalent to maximal Cohen-Macaulay.

For dimension three and greater, there is no analogous classification to the results above. However, we do have partial classifications for higher dimensional rings, two of which we recall below.

Consider the matrix:

$$\left[\begin{array}{c|c|c} x_0^{(1)} \dots x_{n_1-1}^{(1)} & \dots & x_0^{(r)} \dots x_{n_r-1}^{(r)} \\ x_1^{(1)} \dots x_{n_1}^{(1)} & \dots & x_1^{(r)} \dots x_{n_r}^{(r)} \end{array}\right]$$

where each $x_j^{(i)}$ is an indeterminant. We say $R = \mathbb{C}[\![x_0^{(1)},\dots,x_{n_1}^{(1)},\dots,x_0^{(r)},\dots,x_{n_r}^{(r)}]\!]/I$, where I is the ideal generated by the determinants of the 2 by 2 minors of the matrix above, is the *local ring of the scroll of type* $(n_1,\dots,n_r)$, or just R is a *scroll of type* $(n_1,\dots,n_r)$. For scrolls we have the following classification:

Theorem 1.7 ([2])**.** *The local ring of a scroll has finite Cohen-Macaulay representation type if and only if it is type $(m),(1,1),$or $(2,1)$.*

The example below is of particular importance since it closely related to the family of singularities we study in this paper, and it is one of the known dimension three rings of finite Cohen-Macaulay representation type.

Example 1.8 ([2])**.** The type $(2,1)$ scroll is defined as $R = \mathbb{C}[\![u,v,x,y,z]\!]/I$ where I is generated by the maximal minors of the matrix:

$$\begin{bmatrix} z & x & y \\ x & u & v \end{bmatrix}.$$

The complete set of indecomposable non-isomorphic maximal Cohen-Macaulay modules over R is

$$\left\{R, (u,x), (u^2, ux, x^2), (x,y,z), \operatorname{syz}_1^R(u^2, ux, x^2)\right\}.$$

Finally, we have the most well known theorem for classification of rings based on Cohen-Macaulay representation type:

Theorem 1.9 ([15],[4])**.** *The local ring of a hypersurface singularity has finite Cohen-Macaulay representation type if and only if the singularity is ADE.*

Thus, for $R = \mathbb{C}[\![x_1, \ldots x_n]\!]/(f)$, R has finite Cohen-Macaulay representation type if and only if R is isomorphic to the local ring of an ADE hypersurface singularity. This provides an example in every dimension of a ring with finite Cohen-Macaulay representation type. More impressive is the fact that in dimension four and greater these are the only known examples. Based on this result, we would expect the codimension two analogs of the ADE singularities to have finite Cohen-Macaulay type. The remainder shows that in codimension two, the analogs of the ADE singularities have infinite Cohen-Macaulay representation type.

In the next section, we will introduce the analog of the ADE singularities in codimension two, setup the notation for the rest of the paper, and give an outline of the proof that these singularities have infinite Cohen-Macaulay representation type.

1.2 Definitions and Notation

In 2010 Frühbis-Krüger and Neumer published, [10], a classification of simple Cohen-Macaulay codimension two singularities. In their paper, they showed that the family of singularities, over $\mathbb{C}[\![u, v, x, y, z]\!]$, defined by the 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ u & v & f(x,y) \end{bmatrix},$$

where $f(x, y)$ is the defining equation of an ADE curve singularity, behave similarly to the ADE hypersurface singularities. In fact, Frühbis-Krüger and Neumer were able to show that the deformation theory for these singularities is completely determined by the choice of $f(x, y)$. Thus, from a singularity theory perspective, this family of singularities is the analog of the ADE hypersurface singularities in codimension two. To illustrate what we mean by this, we consider the example below. Going forward, we refer to the singularity defined by the 2 by 2 minors of the matrix above as: type $A_k^\sharp$ when $f(x, y) = x^{k+1} + y^2$ $(k \geq 1)$; type $D_k^\sharp$ when $f(x, y) = x^{k-1} + xy^2$ $(k \geq 3)$; type $E_6^\sharp$ when $f(x, y) = x^3 + y^4$; type $E_7^\sharp$ when $f(x, y) = x^3 + xy^3$; and type $E_8^\sharp$ when $f(x, y) = x^3 + y^5$. Similarly, if R is the local ring of an ADE$^\sharp$ singularity, we will identify R by its corresponding singularity type. For example, if R is the local ring of an $E_6^\sharp$ singularity, we say R has type $E_6^\sharp$.

Example 1.10. Consider $f(x, y) = x^3+y^2$, the equation defining the A_2 hypersurface singularity. Let $F(x, y, t) = x^3 + y^2 - tx^2 = x^2(x - t) + y^2$, where t is considered as a parameter. Looking at local rings we have for any non-zero value of t,

$$\mathbb{C}[\![x, y]\!]/F(x, y, t) \cong \mathbb{C}[\![x, y]\!]/(x^2 + y^2),$$

since $(x - t)$ is a unit in $\mathbb{C}[\![x, y]\!]$. Thus, $\mathbb{C}[\![x, y]\!]/F(x, y, t)$ is isomorphic to an A_1 singularity. Hence, we say the A_2 singularity deforms to the A_1 singularity by the deformation $F(x, y, t)$. Now, for the family of singularities above, we have the maximal minors of the matrix,

$$\begin{bmatrix} x & y & z \\ u & v & x^3 + y^2 \end{bmatrix},$$

define the $A_2^\sharp$ singularity. Frühbis-Krüger and Neumer showed that any deformation of the $A_2^\sharp$ can be presented by deforming the defining matrix. Thus, we have

$$\begin{bmatrix} x & y & z \\ u & v & x^3+y^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -tx^2 \end{bmatrix} = \begin{bmatrix} x & y & z \\ u & v & x^2(x-t)+y^2 \end{bmatrix}.$$

Similar to the hypersurface case we have the singularity defined by the minors of the matrix,

$$\begin{bmatrix} x & y & z \\ u & v & x^2(x-t)+y^2 \end{bmatrix},$$

is isomorphic to the $A_1^\sharp$ singularity, when $t \neq 0$. So, we say the $A_2^\sharp$ singularity deforms to the $A_1^\sharp$ singularity, by the deformation

$$\begin{bmatrix} x & y & z \\ u & v & x^2(x-t)+y^2 \end{bmatrix}.$$

Hence, any deformation of $A_2^\sharp$ is determined by a deformation of the A_2 hypersurface singularity. As mentioned above, Frühbis-Krüger and Neumer showed this holds for every singularity in the family.

Furthermore, in the hypersurface case, every ADE singularity deforms to the smooth variety defined by the equation $y^2 = x$. Similarly, for the ADE$^\sharp$ singularities, we have the analog of the smooth variety defined by $y^2 = x$ being expressed by the type $(2,1)$ scroll we saw in Section 1.1, i.e. every ADE$^\sharp$ singularity deforms to the $(2,1)$ scroll.

For the remainder of this paper, S will denote the ring $\mathbb{C}[\![u,v,x,y,z]\!]$ and $\mathscr{F}$ will denote the family of local rings of the singularities over S defined by the maximal minors of the matrix,

$$\begin{bmatrix} x & y & z \\ u & v & f(x,y) \end{bmatrix},$$

where $f(x,y)$ is the defining equation of an ADE curve singularity. We refer to each subfamily of rings by its corresponding type. Furthermore, we use R to denote an arbitrary local ring in $\mathscr{F}$, unless we specify R to be a member of a specific subfamily. The proof of the following property of these rings, which we use throughout the paper, can be found in [10].

Theorem 1.11. *For each R in $\mathscr{F}$, R is a Cohen-Macaulay local ring of dimension 3.*

Furthermore, in [10], Frühbis-Krüger and Neumer showed the singularities we are studying are isolated. Thus, for a ring R in our family, $\mathscr{F}$, we have $R_{\mathfrak{p}}$ is regular, for $\mathfrak{p} \neq (u, v, x, y, z)$ in $\operatorname{Spec} R$. Putting this together, with each R being Cohen-Macaulay, we have Serre's normality criteria holding for each R in $\mathscr{F}$. Thus, we have

Theorem 1.12. *For each R in $\mathscr{F}$, R is a domain.*

As we noted in Section 1.1, there is no broad classification for rings of finite Cohen-Macaulay representation type for dimension three and greater. In dimensions zero, one, and two, maximal Cohen-Macaulay modules have characterizations that make them more accessible. For dimension zero, maximal Cohen-Macaulay is just finitely generated; dimension one, we have equivalence to torsion-free; and dimension two maximal Cohen-Macaulay is equivalent to reflexive. These characterizations not only make it easier to check if a module is maximal Cohen-Macaulay, but it also makes it manageable to write down whole families of modules with these properties. In dimension three we do not have a similar characterization. Notwithstanding that for any ring it is relatively straightforward to write down a family of modules, it is difficult to show that each member of this family is maximal Cohen-Macaulay, and it is also difficult to show that each member of the family is distinct. This is the general outline for the remainder of the paper: present a family of modules over each R in $\mathscr{F}$, show each module in the family is maximal Cohen-Macaulay, and then show each module in the family is distinct.

One of the common themes throughout this thesis will be presenting an ideal or module over each R by only defining the generators of the ideal or module. These generators will remain the same over each ring, R. However, each R in $\mathscr{F}$ has unique relations, so these ideals and modules may behave very differently over each $R \in \mathscr{F}$.

In Chapter 2, we will present three ideals over each R, and prove each is a maximal Cohen-Macaulay module. Then, we will compute the syzygies of one of these ideals, which will be instrumental in a later proof.

In the last chapter, we will present a family of modules over each ring, R. Then, as noted above, we will show each of these is maximal Cohen-Macaulay, and distinct. The proof of maximal Cohen-Macaulay is relatively short. On the other hand, showing that each of the modules in the family is distinct is difficult. The proof of distinctness relies on computing the syzygies of the dual of each member of our family of modules. For this part, we will rely on Singular, [6], to perform the calculations. These calculations with the proof of maximal Cohen-Macaulay will prove the following:

Theorem 1.13. *If R is the local ring of an $ADE^\sharp$ singularity, for $k \leq 2000$ in the A_k and D_k cases, then for each $\alpha \in \mathbb{C}$ the module generated by the column space of the matrix,*

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & ux + \alpha vx \\ 0 & 0 & 0 & x^2 & ux & u^2 \end{bmatrix},$$

is a distinct maximal Cohen-Macaulay module.

As we will show in Section 3.3, this result implies the the main theorem of the paper, found below.

Theorem 1.14. *If R is the local ring of an $ADE^\sharp$ singularity, for $k \leq 2000$ in the A_k and D_k cases, then R has infinitely many indecomposable, non-isomorphic maximal Cohen-Macaulay modules. Hence, R has infinite Cohen-Macaulay representation type.*

Chapter 2

Three Maximal Cohen-Macaulay Ideals

In this chapter, we look at three maximal Cohen-Macaulay ideals over each R in our family of singularities, $\mathscr{F}$. We show these three ideals are, indeed, maximal Cohen-Macaulay in the first section. The second section is devoted to computing the syzygies of one of these ideals; these syzygies are then utilized in Section 3.1.

2.1 Three Maximal Cohen-Macaulay Ideals

Throughout this section, let $\omega_R = (u, x)$, $I = (x, y, z)$, and $\omega_R^2 = (u^2, ux, x^2)$ for each R in $\mathscr{F}$.

Lemma 2.1. *The ideal $\omega_R = (u, x)$ is a maximal Cohen-Macaulay ideal over each $R \in \mathscr{F}$.*

Proof. For ω_R, we have $\bar{R} := R/\omega_R$ is isomorphic to the hypersurface defined by $yf(0, y) - vz$ in $\mathbb{C}[\![v, y, z]\!]$. Thus, we have the dimension of $\bar{R}$ is 2. Furthermore, we have $\bar{R}$ is a complete intersection, and hence a Cohen-Macaulay ring. Therefore, the depth of $\bar{R}$ is 2. Now consider the short exact sequence:

$$0 \longrightarrow \omega_R \longrightarrow R \longrightarrow \bar{R} \longrightarrow 0.$$

From the long exact sequence of Ext, we have the following section of the sequence:

$$\ldots \longrightarrow \operatorname{Ext}^1_R(\mathbb{C}, \bar{R}) \longrightarrow \operatorname{Ext}^2_R(\mathbb{C}, \omega_R) \longrightarrow \operatorname{Ext}^2_R(\mathbb{C}, R) \longrightarrow \ldots$$

Since the depth of $\bar{R}$ is 2, we have $\operatorname{Ext}^1_R(\mathbb{C}, \bar{R}) = 0$. Similarly, we have $\operatorname{Ext}^2_R(\mathbb{C}, R) = 0$. Thus, we have $\operatorname{Ext}^2_R(\mathbb{C}, \omega_R) = 0$, and so depth $\omega_R > 2$. However, since ω_R is an ideal of R we have $\operatorname{depth}\omega_R \leq \operatorname{depth} R = 3$. Therefore, $\operatorname{depth}\omega_R = 3$. As a result, we have ω_R is a maximal Cohen-Macaulay module. $\square$

It is known that ω_R is, in fact, the dualizing module for each R. However, we disregard this for the purposes of this work. Now we consider the ideal I.

Lemma 2.2. *$I = (x, y, z)$ is a maximal Cohen-Macaulay ideal over each $R \in \mathscr{F}$.*

Proof. The quotient of R by I is isomorphic to the ring $\mathbb{C}[\![u, v]\!]$. This is a regular ring, and hence a Cohen-Macaulay ring of dimension 2. Consider the short exact sequence below:

$$0 \longrightarrow I \longrightarrow R \longrightarrow \mathbb{C}[\![u, v]\!] \longrightarrow 0$$

From this we get the following section of the long exact sequence of Ext:

$$\ldots \longrightarrow \operatorname{Ext}^1_R(\mathbb{C}, \mathbb{C}[\![u, v]\!]) \longrightarrow \operatorname{Ext}^2_R(\mathbb{C}, I) \longrightarrow \operatorname{Ext}^2_R(\mathbb{C}, R) \longrightarrow \ldots.$$

The depth of $\mathbb{C}[\![u, v]\!]$ is two, since it is a Cohen-Macaulay ring. Hence, we have $\operatorname{Ext}^1_R(\mathbb{C}, \mathbb{C}[\![u, v]\!]) = 0$. Furthermore, we have that the depth of R is 3, and so $\operatorname{Ext}^2_R(\mathbb{C}, R) = 0$. Thus, we have $\operatorname{Ext}^2_R(\mathbb{C}, I) = 0$, and so the depth of I is at least 3. Finally, since I is an ideal of R the depth of I is at most 3, and so we have $\operatorname{depth} I = 3$. Therefore, I is a maximal Cohen-Macaulay module. $\square$

For the proof that ω_R^2 is maximal Cohen-Macaulay, we will be using Singular, [6], to compute the depth of R/ω_R^2; and so we will delay this until the end of the next section after we are more familiar with computations in Singular.

2.2 The Syzygies of ω_R^2

In this section, we compute the syzygies of the ideal $\omega_R^2 = (x^2, ux, u^2)$, and show this ideal is maximal Cohen-Macaulay. These syzygies will be crucial in the proof that the family of modules, we define in Chapter 3, is maximal Cohen-Macaulay. The form of the syzygies depends on the type of R. However, each set of syzygies is very similar from one singularity to the next. After stating the main theorem of this section, we will prove two of the cases, and leave the other three cases to the interested reader.

Anytime we compute a standard basis we will be using the negative graded reverse lexicographical ordering. For those unfamiliar with standard bases, which are the analog of Gröbner bases for local rings, the texts [12] and [5] both provide a great introduction to the topic. An ideal in a power series ring may have terms of unbounded degree. Hence, the usual monomial orderings we consider in polynomial rings do not produce a well-defined definition of leading term. To fix this, we consider the leading terms to be those of least degree for an ideal in a power series ring. For example, with the negative graded reverse lexicographical ordering, we have the following monomial orderings:

- $1 > u, v, x, y, z > u^2, v^2, x^2, y^2, z^2 > \ldots$.
- $u > v > x > y > z$.
- $vx > uy$.

The algorithm for computing a standard basis is similar to Buchberger's algorithm for computing a Gröbner basis. However, in place of using polynomial division, the algorithm for a standard basis applies Mora's normal form algorithm to ensure that Buchberger's algorithm halts.

We will be using Singular to compute a standard basis and the syzygies of this basis. In Singular, we simply declare the characteristic of the field we want to work over. Since our singularities are defined over $\mathbb{C}$ we will use a characteristic 0 field. Looking at the modified Buchberger's algorithm for standard bases, [12] or [5], it should be clear that for an ideal or module over a ring with coefficients in a ring or field, K, a standard basis for that ideal or module will also have coefficients in the same ring or field, K. Since all of the ideals and modules we look at have rational coefficients, the coefficients of elements in a standard basis will remain rational. The syzygies are a by-product of computing a standard basis by Schreyer's Theorem, Chapter 5, Section 4 of [5]. Thus, any syzygy modules we compute will have rational coefficients. Therefore, any computation we do using Singular would be the same over $\mathbb{C}$. Now we state the main theorem of this section.

Theorem 2.3. *The module of syzygies of the ideal $\omega_R^2 = (x^2, ux, u^2)$ is generated by the columns of the matrix below:*

$$\begin{bmatrix} u & v & f(x,y) & 0 & 0 & 0 \\ -x & -y & -z & u & v & f(x,y) \\ 0 & 0 & 0 & -x & -y & -z \end{bmatrix},$$

where

$$f(x,y) = \begin{cases} x^{k+1} + y^2 & R \text{ is type } A_k^\sharp \\ x^{k-1} + xy^2 & R \text{ is type } D_k^\sharp \\ x^3 + y^4 & R \text{ is type } E_6^\sharp \\ x^3 + xy^3 & R \text{ is type } E_7^\sharp \\ x^3 + y^5 & R \text{ is type } E_8^\sharp. \end{cases}$$

Before proving Theorem 2.3, we prove the lemma below. To justify the need for this lemma, let $J \subset S = \mathbb{C}[\![u, v, x, y, z]\!]$ be the defining ideal for an R in our family, $\mathscr{F}$, and $\phi : S \to S/J = R$. Then, $(a_1, a_2, a_3)^T \in \operatorname{syz}_1^R \omega_R^2$ if and only if $a_1x^2 + a_2ux + a_3u^2 = 0$ in R. This is equivalent to $a_1x^2 + a_2ux + a_3u^2 \in J$ in S. All such (a_1, a_2, a_3) must arise from a syzygy of the S-ideal $(x^2, ux, u^2) + J$. Thus, to find all such (a_1, a_2, a_3), we consider the inverse image ideal, $\phi^{-1}(\omega_R^2) = (x^2, ux, u^2) + J$, of ω_R^2 in the ring S by ϕ. We compute a generating set for the syzygies of $\phi^{-1}(\omega_R^2)$ in S. Then, after modding out by the ideal J, we have a generating set for $\operatorname{syz}_1^R \omega_R^2$. For the remainder of this section, let ϕ be defined as above. After completing the proof for the $A_k^\sharp$ case, we will prove a similar lemma and theorem for the $D_k^\sharp$ case.

Lemma 2.4. *Let R be the coordinate ring of a type $A_k^\sharp$ singularity and $\overline{\omega_R^2} = (f_1 := x^2, f_2 := ux, f_3 := u^2, f_4 := vx - uy f_5 := x^{k+2} + xy^2 - uz, f_6 := x^{k+1}y + y^3 - vz)$ be the inverse image of the ideal ω_R^2 in S by ϕ. Then, a standard basis for $\overline{\omega_R^2}$ in S is $G = \{u^2, ux, vx - uy, x^2, uz - xy^2, vz - y^3\}$.*

Proof. First, notice that $\overline{\omega_R^2} = (u^2, ux, vx - uy, x^2, uz - xy^2, vz - y^3)$, since $k \geq 1$, $x^k(x^2) - (uz - xy^2) = x^{k+2} + xy^2 - uz$, and $x^{k-1}y(x^2) - (vz - y^3) = x^{k+1}y + y^3 - vz$. Now that the generators of $\overline{\omega_R^2}$ do not involve an arbitrary power we use Singular to compute a standard basis of $\overline{\omega_R^2}$ with this set of generators. Let $G = \{g_1 := u^2, g_2 := ux, g_3 := vx - uy, g_4 := x^2, g_5 := uz - xy^2, g_6 := vz - y^3\}$.

We can run the following Singular code to compute a standard basis for G in S with the negative graded reverse lexicographical ordering:

```
ring S=0,(u,v,x,y,z),ds;
ideal w2bar=u2,ux,vx-uy,x2,uz-xy2,vz-y3;
ideal stdw2bar=std(w2bar);
print(stdw2bar);
```

Figure 2.1 Standard basis for $\overline{\omega_R^2}$ using Singular.

This returns the same set G, and so the set G is a standard basis, which generates $\overline{\omega_R^2}$. Thus, G is a standard basis for $\overline{\omega_R^2}$. □

Now, using Singular, we can compute the syzygies of G, by the following code.

```
matrix syzG = syz(stdw2);
print(syzG);
```

Figure 2.2 Syzygies of G using Singular.

Letting $G = \{g_1 = x^2, g_2 = ux, g_3 = u^2, g_4 = vx - uy, g_5 = uz - xy^2, g_6 = vz - y^3\}$ and reordering the columns, we have the module of syzygies of G are generated by the column space of the matrix below.

$$\begin{bmatrix} u & v & y^2 & 0 & 0 & 0 & 0 & 0 \\ -x & -y & -z & u & -v & y^2 & 0 & 0 \\ 0 & 0 & 0 & -x & y & -z & 0 & 0 \\ 0 & -x & 0 & 0 & u & 0 & -y^2 & -z \\ 0 & 0 & x & 0 & 0 & u & -v & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & u & x \end{bmatrix}$$

Now we can prove Theorem 2.3 for the type $A_k^\sharp$ case.

Proof. From the discussion above, we have a generating set for the syzygies of G over the ring, S. Now we rewrite the syzygies of the g_i's in terms of the f_i's, which will be the syzygies of $\overline{\omega_R^2}$. First, note that we have the following relations between the g_i's and f_i's:

$$\begin{aligned} g_1 &= f_1 & \qquad g_2 &= f_2 \\ g_3 &= f_3 & g_4 &= f_4 \\ g_5 &= x^k f_1 - f_5 & g_6 &= x^{k-1} y f_1 - f_6. \end{aligned}$$

Listing the syzygies of $\overline{\omega_R^2}$, we have:

$$\begin{aligned} s_1 &= ug_1 - xg_2 && \longrightarrow uf_1 - xf_2 \\ s_2 &= vg_1 - yg_2 - xg_4 && \longrightarrow vf_1 - yf_2 - xf_4 \\ s_3 &= y^2 g_1 - zg_2 + xg_5 && \longrightarrow (x^{k+1} + y^2) f_1 - zf_2 - xf_5 \\ s_4 &= ug_2 - xg_3 && \longrightarrow uf_2 - xf_3 \\ s_5 &= -vg_2 + yg_3 + xg_4 && \longrightarrow -vf_2 + yf_3 + xf_4 \\ s_6 &= y^2 g_2 - zg_3 + ug_5 && \longrightarrow ux^k f_1 + y^2 f_2 - zf_3 - uf_5 \\ s_7 &= -y^2 g_4 - vg_5 + ug_6 && \longrightarrow x^{k-1}(uy - vx) f_1 - y^2 f_4 + vf_5 - uf_6 \\ s_8 &= -zg_4 - yg_5 + xg_6 && \longrightarrow -zf_4 + yf_5 - xf_6. \end{aligned}$$

Hence, we have the syzygies of $\overline{\omega_R^2}$ are generated by the columns of the matrix:

$$\begin{bmatrix} u & v & x^{k+1}+y^2 & 0 & 0 & ux^k & x^{k-1}(uy-vx) & 0 \\ -x & -y & -z & u & -v & y^2 & 0 & 0 \\ 0 & 0 & 0 & -x & y & -z & 0 & 0 \\ 0 & -x & 0 & 0 & x & 0 & -y^2 & -z \\ 0 & 0 & -x & 0 & 0 & -u & v & y \\ 0 & 0 & 0 & 0 & 0 & 0 & -u & -x \end{bmatrix}.$$

Note that any relation on the generators of $\overline{\omega_R^2}$ will produce a syzygy of ω_R^2 in R. Furthermore, any syzygy of ω_R^2 in R must arise in this way. Thus, considering the matrix above in R, by using that $xv - uy = x^{k+2} + xy^2 - uz = x^{k+1}y + y^3 - vz = 0$ in R, we get the matrix:

$$\begin{bmatrix} u & v & x^{k+1}+y^2 & 0 & 0 & ux^k & 0 & 0 \\ -x & -y & -z & u & -v & y^2 & 0 & 0 \\ 0 & 0 & 0 & -x & y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By moving columns around, applying negatives to certain columns, and removing extraneous rows and columns, we get the syzygy matrix:

$$\begin{bmatrix} u & v & x^{k+1}+y^2 & 0 & 0 & ux^k \\ -x & -y & -z & u & v & y^2 \\ 0 & 0 & 0 & -x & -y & -z \end{bmatrix}.$$

Relabeling these as $s_1, \dots s_6$ respectively, we have

$$s_6 - x^k s_1 = (0, x^{k+1} + y^2, -z)^T,$$

and so we can replace s_6 by $(0, x^{k+1} + y^2, -z)^T$. This gives us the claimed generating set for $\operatorname{syz}_1^R \omega_R^2$. $\square$

Now we turn to the $D_k^\sharp$ case.

Lemma 2.5. *Let R be the coordinate ring of a type $D_k^\sharp$ singularity and $\overline{\omega_R^2} = (f_1 := x^2, f_2 := ux, f_3 := u^2, f_4 := vx - uy, f_5 := x^k + x^2y^2 - uz, f_6 := x^{k-1}y + xy^3 - vz)$ be the inverse image of the ideal ω_R^2 in S by ϕ. Then, a standard basis for $\overline{\omega_R^2}$ in S is $G = \{u^2, ux, vx - uy, x^2, uz, vz - xy^3\}$.*

Proof. We start by taking a different generating set for the ideal. We have $\overline{\omega_R^2} = (u^2, ux, vx - uy, x^2, uz, vz - xy^3)$, since $x^k + x^2y^2 - uz = (x^{k-2} + y^2)(x^2) - uz$ and $x^{k-1}y + xy^3 - vz = x^{k-3}y(x^2) - (vz - xy^3)$. Let $G = \{g_1 := u^2, g_2 := ux, g_3 := vx - uy, g_4 := x^2, g_5 := uz, g_6 := vz - xy^3\}$. Since G does not involve any arbitrary exponents, we can, again, use the following Singular code to check that G is indeed a standard basis.

```
ring S=0,(u,v,x,y,z),ds;
ideal w2bar=u2,ux,vx-uy,x2,uz,vz-xy3;
ideal stdw2bar=std(w2bar);
print(stdw2bar);
```

Figure 2.3 Standard basis for $\overline{\omega_R^2}$ using Singular.

This returns the same set G, and so G is a standard basis. Hence, G is a standard basis which generates $\overline{\omega_R^2}$, and so G is a standard basis of $\overline{\omega_R^2}$. □

Once more, we can use Singular to compute the syzygies of G using similar lines of code as we did for the $A_k^\sharp$ case.

We relabel the g_i's as $G = \{g_1 = x^2, g_2 = ux, g_3 = u^2, g_4 = vx - uy, g_5 = uz, g_6 = vz - xy^3\}$. Reordering the columns, with respect to this relabeling, we have the module of syzygies of G is generated by the column space of the matrix below. Using this generating set for the syzygies of G, we find a generating set for the syzygies of ω_R^2 in the $D_k^\sharp$ case; proving Theorem 2.3 in the $D_k^\sharp$ case.

$$\begin{bmatrix} u & v & 0 & 0 & 0 & 0 & y^3 & 0 \\ -x & -y & -z & u & -v & 0 & 0 & y^3 \\ 0 & 0 & 0 & -x & y & -z & 0 & 0 \\ 0 & -x & 0 & 0 & u & 0 & z & 0 \\ 0 & 0 & x & 0 & 0 & u & -y & -v \\ 0 & 0 & 0 & 0 & 0 & 0 & x & u \end{bmatrix}.$$

Proof. From the discussion above, the columns of the matrix above form a generating set for syzygies of G, which is a standard basis for $\overline{\omega_R^2}$ over S. We use this generating set to find a generating set for the syzygies of $\overline{\omega_R^2}$ over S. Note we have the following relations for the g_i's in terms of the f_i's:

$$\begin{aligned} g_1 &= f_1 & \quad g_2 &= f_2 \\ g_3 &= f_3 & \quad g_4 &= f_4 \\ g_5 &= (x^{k-2}+y^2)f_1 - f_5 & \quad g_6 &= x^{k-3}f_1 - f_6 \end{aligned}$$

We substitute these relations into the syzygies, $s_1, \ldots, s_8$, from the matrix above.

$$\begin{aligned} s_1 &= ug_1 - xg_2 && \longrightarrow && uf_1 - xf_2 \\ s_2 &= vg_1 - yg_2 - xg_4 && \longrightarrow && vf_1 - yf_2 - xf_4 \\ s_3 &= -zg_2 + xg_5 && \longrightarrow && (x^{k-1}+xy^2)f_1 - zf_2 - xf_5 \\ s_4 &= ug_2 - xg_3 && \longrightarrow && uf_2 - xf_3 \\ s_5 &= -vg_2 + yg_3 + ug_4 && \longrightarrow && -vf_2 + yf_3 + uf_4 \\ s_6 &= -zg_3 + ug_5 && \longrightarrow && u(x^{k-2}+y^2)f_1 - zf_3 - uf_5 \\ s_7 &= y^3g_1 + zg_4 - yg_5 + xg_6 && \longrightarrow && zf_4 + yf_5 - xf_6 \\ s_8 &= y^3g_2 - vg_5 + ug_6 && \longrightarrow && (x^{k-3}(uy-vx) - vy^2)f_1 + y^3f_2 + vf_5 - uf_6 \end{aligned}$$

In R, we have $f_4 = f_5 = f_6 = 0$. Thus writing $s_1, \ldots, s_8$ as a matrix in R, the syzygies of ω_R^2 are generated by the column space of the matrix:

$$\begin{bmatrix} u & v & x^{k-1}+xy^2 & 0 & 0 & u(x^{k-2}+y^2) & 0 & -vy^2 \\ -x & -y & -z & u & -v & 0 & 0 & y^3 \\ 0 & 0 & 0 & -x & y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Deleting trivial rows and columns and multiplying by -1 to some columns we get the following matrix:

$$\begin{bmatrix} u & v & x^{k-1}+xy^2 & 0 & 0 & u(x^{k-2}+y^2) & -vy^2 \\ -x & -y & -z & u & v & 0 & y^3 \\ 0 & 0 & 0 & -x & -y & -z & 0 \end{bmatrix}.$$

Relabeling the columns above as $s_1, \ldots, s_7$, respectively, we have $s_7 = -y^2 s_2$. So we can delete s_7 from our generating set. Furthermore, we have

$$s_6 - (x^{k-2}+y^2)s_1 = (0, x^{k-1}+xy^2, -z)^T,$$

and so we can replace s_6 by $(0, x^{k-1}+xy^2, -z)^T$ in our generating set. This gives the claimed generating set for $\operatorname{syz}_1^R \omega_R^2$ in the $D_k^\sharp$ case. $\square$

As noted at the beginning of this section, similar methods can be used to find the syzygy matrix for ω_R^2 in the $E_6^\sharp, E_7^\sharp$, and $E_8^\sharp$ cases.

Finally, as mentioned at the end of the Section 1.2, we will use Singular to help show the depth of ω_R^2 is 3, for each $R \in \mathscr{F}$.

Lemma 2.6. *ω_R^2 is a maximal Cohen-Macaulay ideal over each $R \in \mathscr{F}$.*

Proof. First, let R be the local ring of a type $A_k^\sharp$ singularity. Let $\overline{\omega_R^2}$ be the inverse image of the ideal ω_R^2 in the ring S by ϕ. This gives us

$$R/\omega_R^2 \cong S/\overline{\omega_R^2} = \mathbb{C}[\![u,v,x,y,z]\!]/(u^2, ux, vx-uy, x^2, uz-xy^2, vz-y^3).$$

Since R/ω_R^2 does not involve an arbitrary power of x, we can employ Singular to compute the depth of it. Running the following code on Singular returns the depth of R/ω_R^2 as 2.

```
LIB "homolog.lib";
ring S=0,(u,v,x,y,z),ds;
ideal w2bar=u2,ux,vx-uy,x2,uz-xy2,vz-y3;
qring Rbar = std(w2bar);
module Z=0;
depth(Z);
```

Figure 2.4 Singular code to find depth R/ω_R^2 in the $A_k^\sharp$ case.

Notice that in Singular the function depth returns the depth of the cokernel of the module in the depth function. Now that the depth of R/ω_R^2 is 2, the proof that ω_R^2 has depth 3, follows similarly to lemmas 2.1, and 2.2. Therefore, ω_R^2 is a maximal Cohen-Macaulay module over R in the $A_k^\sharp$ case.

Now let R be the local ring of a $D_k^\sharp$ singularity, and $\overline{\omega_R^2}$ be the inverse image of the ideal ω_R^2 in S by ϕ. This gives us

$$R/\omega_R^2 \cong S/\overline{\omega_R^2} = \mathbb{C}[\![u,v,x,y,z]\!]/(u^2, ux, vx-uy, x^2, uz, vz-xy^3).$$

Again, having removed any arbitrary powers of x, we can run similar Singular code to check that depth $R/\omega_R^2 = 2$. Then, as in the $A_k^\sharp$ case, we have ω_R^2 has depth 3, and hence is maximal Cohen-Macaulay. Finally, for the $E_6^\sharp$, $E_7^\sharp$, and $E_8^\sharp$ cases, we can use Singular to show the depth of ω_R^2 is 3.

Therefore, for any R in our family, $\mathscr{F}$, we have depth $\omega_R^2 = 3$. So over each $R \in \mathscr{F}$, we have ω_R^2 is maximal Cohen-Macaulay. $\square$

Chapter 3

Main Theorem

In this chapter, we will prove that the family of rings defined by the maximal minors of the matrix:

$$\begin{bmatrix} x & y & z \\ u & v & f(x,y) \end{bmatrix},$$

where $f(x,y)$ is $x^{k+1}+y^2$, $x^{k-1}+xy^2$, x^3+y^4, x^3+xy^3, or x^3+y^5 has infinite Cohen-Macaulay representation type. To accomplish this, we will construct an infinite family of rank two maximal Cohen-Macaulay modules over each singularity. Fortunately, the family of modules, $\mathscr{M}_t$, over each ring is generated by the same generators. For each R, let $\mathscr{M}_t$ be the module generated by the columns of the matrix:

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & xu+txv \\ 0 & 0 & 0 & x^2 & xu & u^2 \end{bmatrix},$$

where we are considering t as the parameter of the family, i.e. $t \in \mathbb{C}$. Hence, some of the time we will be thinking of $\mathscr{M}_t$ as a family of modules, where each $t \in \mathbb{C}$, determines a member of the family. While other times, we will be thinking of $\mathscr{M}_t$ as a single module over the polynomial ring $R[t]$ with coefficients in R and variable t. The context should make which case we are considering clear. Throughout this chapter, $\mathscr{M}_\alpha$ will denote the module generated by columns of the matrix:

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & xu+\alpha xv \\ 0 & 0 & 0 & x^2 & xu & u^2 \end{bmatrix},$$

over a ring R from our family, $\mathscr{F}$.

We will break the proof of the main theorem into two parts. First, we show the modules defined above are maximal Cohen-Macaulay and rank two. This will be relatively quick, since we have done most of this work in Section 2.2, and we can do this without breaking up the proof over each ring. Section 3.2 is dedicated to justifying the calculations in Section 3.3. The second half of the proof, showing each module of $\mathscr{M}_t$ is distinct, is the content of Section 3.3.

3.1 Proof of Maximal Cohen-Macaulayness

In view of the fact that the family of modules we are discussing involves a parameter, t, we will work in the ring $S[t] = \mathbb{C}[\![u, v, x, y, z]\!][t]$ and the corresponding rings, $R[t]$, where R is a local ring from our family, $\mathscr{F}$.

Theorem 3.1. *Let $I = (x, y, z)$ and $\omega_R^2 = (x^2, ux, u^2)$ be the maximal Cohen-Macaulay ideals over R from Chapter 2. Furthermore, let $\mathscr{M}_t$ be the $R[t]$-module generated by the columns of the matrix*

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & xu + txv \\ 0 & 0 & 0 & x^2 & xu & u^2 \end{bmatrix}.$$

Then the following sequence:

$$0 \longrightarrow xI[t] \xrightarrow{\psi} \mathscr{M}_t \xrightarrow{\varphi} \omega_R^2[t] \longrightarrow 0,$$

is exact, where ψ is inclusion in the first component and φ is projection from the second component.

Proof. Let $u_1, \ldots, u_6$ be the generators of $\mathscr{M}_t$, respectively. First, note that $\varphi \circ \psi = 0$, i.e. $\operatorname{im} \psi \subset \ker \varphi$. For the other inclusion, consider $a = a_4u_4 + a_5u_5 + a_6u_6 \in \ker \varphi$. If $a_6 = 0$, then $\varphi(a_4u_4 + a_5u_5) = 0$ implies $a_4x^2 + a_5ux = 0$, and so $a = a_4u4 + a_5u_5 = 0$; hence $a \in \operatorname{im} \psi$. Next, assume $a_6 \neq 0$. Since $a \in \ker \varphi$, we have

$$a_4x^2 + a_5ux + a_6u^2 = 0.$$

Hence, $(a_4, a_5, a_6)^T \in \mathrm{syz}_1(\omega_R^2[t])$. Note that $\omega_R^2[t]$ is a $R[t]$-module by considering elements of $\omega_R^2[t]$ as polynomials in the variable t with coefficients in ω_R^2. Then, we have that $\omega_R^2[t] \cong R[t] \otimes_R \omega_R^2$; and since $R[t]$ is flat over R, the set of syzygies for ω_R^2 over R remain the complete set of syzygies for ω_R^2 as a $R[t]$-module. Recall from Section 2.2, we have $\mathrm{syz}_1^R \omega_R^2$ is the column space of the matrix,

$$\begin{bmatrix} u & v & f(x,y) & 0 & 0 & 0 \\ -x & -y & -z & u & v & f(x,y) \\ 0 & 0 & 0 & -x & -y & -z \end{bmatrix},$$

where $f(x,y)$ is the corresponding defining equation for R. Thus, $a_6 \in I[t] = (x,y,z)R[t]$. Hence, we have

$$a_6(ux + tvx) = (u + tv)xa_6 \in x(x,y,z)R[t] = xI[t],$$

i.e. $a_6(ux + tvx) \in xI[t]$. Hence, if $a \in \ker \varphi$, we have $a = (r, 0)^T$ for some $r \in xI[t]$. Thus, $a \in \operatorname{im} \psi$. $\square$

Corollary 3.2. *The $R[t]$-modules $\omega_R^2[t]$, $xI[t]$, and $\mathscr{M}_t$, as defined above, are Cohen-Macaulay. Furthermore, we have that* $\mathrm{depth}(\omega_R^2[t])_{\mathfrak{m}} = \mathrm{depth}(xI[t])_{\mathfrak{m}} = \mathrm{depth}(\mathscr{M}_t)_{\mathfrak{m}}$, *for any maximal ideal $\mathfrak{m} \subset R[t]$.*

Proof. First, notice that x is regular on I, since R is a domain. Thus, the depth of xI is 3 in R. We then have $\omega_R^2[t]$ and $xI[t]$ are Cohen-Macaulay at any maximal ideal of $R[t]$, by Theorem 2.1.9 in [3]. Next, since localization is an exact functor, the short exact sequence from Theorem 3.1 will stay exact when localized at any maximal ideal, $\mathfrak{m} \subset R[t]$. Hence, at each maximal ideal, $\mathfrak{m}$, we have $\mathrm{depth}(\mathscr{M}_t)_{\mathfrak{m}} = \mathrm{depth}(xI[t])_{\mathfrak{m}} = \mathrm{depth}(\omega_R^2[t])_{\mathfrak{m}}$, by the long exact sequence of Ext. This implies both parts of the corollary. $\square$

Corollary 3.3. *For $\alpha \in \mathbb{C}$, $\mathscr{M}_\alpha$ is maximal Cohen-Macaulay over R, for each R in $\mathscr{F}$.*

Proof. For each $\alpha \in \mathbb{C}$, we have $t-\alpha$ is regular on $R[t]$, since $R[t]$ is a domain. This implies that $t-\alpha$ is regular on $R[t]^2$. Since $\mathscr{M}_t$ is a submodule of $R[t]^2$, we have $t-\alpha$ is regular on $\mathscr{M}_t$. Thus, $\mathscr{M}_\alpha = \mathscr{M}_t/(t-\alpha)\mathscr{M}_t$ is a Cohen-Macaulay R-module, by Theorem 2.1.3 of [3]. To see it is maximal Cohen-Macaulay, note that $t-\alpha$ is regular on $xI[t]$ and $\omega_R^2[t]$. Let $\mathfrak{m}_\alpha = (t-\alpha)+\mathfrak{n} \subset R[t]$, where $\mathfrak{n}$ is the maximal ideal of R. Clearly, $\mathfrak{m}_\alpha$ is a maximal ideal of $R[t]$, since $R[t]/\mathfrak{m}_\alpha \cong \mathbb{C}$. For any maximal regular sequence, $\underline{l} \subset \mathfrak{n}$, of xI or ω_R^2 in R, we have $(t-\alpha, \underline{l})$ is a maximal regular sequence for $xI[t]$ or $\omega_R^2[t]$ in $\mathfrak{m}_\alpha$. Therefore, $\operatorname{depth}(xI[t])_{\mathfrak{m}_\alpha} = \operatorname{depth}(\omega_R^2[t])_{\mathfrak{m}_\alpha} = 4$ in $(R[t])_{\mathfrak{m}_\alpha}$. By the previous corollary, this implies that $\operatorname{depth}(\mathscr{M}_t)_{\mathfrak{m}_\alpha} = 4$. Thus, we have $\operatorname{depth} \mathscr{M}_\alpha = 3$ in R, by Theorem 1.2.10 in [3]. Hence, for each $\alpha \in \mathbb{C}$, $\mathscr{M}_\alpha$ is maximal Cohen-Macaulay, for each $R \in \mathscr{F}$. $\square$

Finally, we show that each module in $\mathscr{M}_t$ is rank two over each $R \in \mathscr{F}$. Similar to the previous two corollaries, this follows from Theorem 3.1. The fact that, for each $\alpha \in \mathbb{C}$, $\mathscr{M}_\alpha$ is rank two will be used in Section 3.3 to prove that each $R \in \mathscr{F}$ has infinite Cohen-Macaulay type.

Corollary 3.4. *For each $\alpha \in \mathbb{C}$ and each $R \in \mathscr{F}$, $\mathscr{M}_\alpha$ is a rank two R-module.*

Proof. First, note that, by the inclusion $\mathbb{C}[t] \hookrightarrow R[t]$, we have $xI[t]$, $\omega_R^2[t]$, and $\mathscr{M}_t$ are $\mathbb{C}[t]$-modules. Let $k(\alpha) = \mathbb{C}[t]/(t-\alpha)$. Then we have $xI[t] \otimes_{\mathbb{C}[t]} k(\alpha) \cong xI$, $\omega_R^2[t] \otimes_{\mathbb{C}[t]} k(\alpha) \cong \omega_R^2$, and $\mathscr{M}_t \otimes_{\mathbb{C}[t]} k(\alpha) \cong \mathscr{M}_\alpha$ as R-modules. Thus, tensoring the short exact sequence from Theorem 3.1 with $k(\alpha)$, we get the complex:

$$\ldots \to \operatorname{Tor}_1^{\mathbb{C}[t]}(\omega_R^2[t], k(\alpha)) \to xI \to \mathscr{M}_\alpha \to \omega_R^2 \to 0.$$

The sequence:

$$0 \longrightarrow xI \longrightarrow \mathscr{M}_\alpha \longrightarrow \omega_R^2 \longrightarrow 0$$

is exact if $\operatorname{Tor}_1^{\mathbb{C}[t]}(\omega_R^2[t], k(\alpha)) = 0$. To compute this Tor, let $\mathbb{K}(t-\alpha)$ be the Koszul complex over $\mathbb{C}[t]$ of the single element $t-\alpha$. Then, $\mathbb{K}(t-\alpha)$ is a resolution of $k(\alpha)$

as a $\mathbb{C}[t]$-module. Tensoring $\mathbb{K}(t-\alpha)$ with $\omega_R^2[t]$, we have $\operatorname{Tor}_1^{\mathbb{C}[t]}(\omega_R^2[t], k(\alpha))$ is the first homology of this tensored complex. However, $t-\alpha$ is regular on $\omega_R^2[t]$, since $R[t]$ is a domain. Thus, $\mathbb{K}(t-\alpha)\otimes_{\mathbb{C}[t]}\omega_R^2[t]$ is exact. Therefore, all the homology modules of $\mathbb{K}(t-\alpha)\otimes_{\mathbb{C}[t]}\omega_R^2[t]$ are zero; and so $\operatorname{Tor}_1^{\mathbb{C}[t]}(\omega_R^2[t], k(\alpha)) = 0$. Consequently, we have for each $\alpha \in \mathbb{C}$ and each $R \in \mathscr{F}$, the complex

$$0 \longrightarrow xI \longrightarrow \mathscr{M}_\alpha \longrightarrow \omega_R^2 \longrightarrow 0$$

is a short exact sequence. Since R is a domain, we have any nonzero ideal of R has rank one as an R-module. Furthermore, since R is a domain, we have rank is additive on short exact sequences. This implies the rank of $\mathscr{M}_\alpha$ is two. □

3.2 Specialization

We now begin the process of showing each member of $\mathscr{M}_t$ is distinct. Recall throughout this chapter, we let $\mathscr{M}_\alpha$ denote the module generated by the column space of the matrix:

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & ux+\alpha vx \\ 0 & 0 & 0 & x^2 & ux & u^2 \end{bmatrix},$$

where $\alpha \in \mathbb{C}$. In this section, we will be using the following isomorphic forms of the module $\mathscr{M}_\alpha$ at different times:

$$\mathscr{M}_\alpha \cong \mathscr{M}_t/(t-\alpha)\mathscr{M}_t \cong \mathscr{M}_t \otimes_R R[t]/(t-\alpha) \cong \mathscr{M}_t \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t-\alpha).$$

The first isomorphism is obvious and the second isomorphism is well known. The third follows from $R[t]$ being a $\mathbb{C}[t]$-algebra, and the natural inclusion $\mathbb{C}[t] \hookrightarrow R[t]$.

Recall the following definition:

Definition 3.5. Let M be a finitely generated R-module, with generators $m_1, \ldots, m_n$ and relations

$$a_{j1}m_1 + \ldots a_{jn}m_n = 0, \text{ for } j = 1, \ldots$$

Then the *ith Fitting ideal,* $\mathrm{Fitt}_i(M)$, is the ideal generated by the $n-i$ minors of (a_{jk}).

Accordingly, the ith Fitting ideal of a module M is the ideal generated by the $n-i$ minors of a presentation matrix for M. It is well-known that these ideals do not depend on the choice of presentation and that isomorphic modules must have the same Fitting ideals, see Section 20.2 in [7]. Consequently, in our case, we will be using the fact that if two modules have different Fitting ideals, then they are non-isomorphic.

To show that for each $\alpha \in \mathbb{C}$ we get a distinct $\mathscr{M}_\alpha$ over R, we compute a Fitting ideal of $\mathrm{Hom}_R(\mathscr{M}_\alpha, R)$, for each $\alpha \in \mathbb{C}$. Throughout this section, for a module M over R (respectively, $R[t]$), we will let M^* denote the dual module, $\mathrm{Hom}_R(M, R)$ (respectively, $\mathrm{Hom}_{R[t]}(M, R[t])$), of the module M. As we mentioned in Chapter 1, computing the Fitting ideals will require the use of Singular. We will present the Singular code to compute the Fitting ideal we need, and then present the code to check this for $A_k^\sharp$ and $D_k^\sharp$ for $k \leq 2000$. However, in order to do this we have to perform these calculations over $R[t]$ for the module $\mathscr{M}_t$, and then specialize to the module $\mathscr{M}_\alpha$, for each $\alpha \in \mathbb{C}$. There are three lemmas we must address to justify this process:

- $\mathrm{syz}_1^{R[t]}\left(\mathscr{M}_t\right)/(t-\alpha)\,\mathrm{syz}_1^{R[t]}\left(\mathscr{M}_t\right) \cong \mathrm{syz}_1^R(\mathscr{M}_\alpha)$ for each $\alpha \in \mathbb{C}$.
- $\mathscr{M}_t^*/(t-\alpha)\mathscr{M}_t^* \cong \mathscr{M}_\alpha^*$ for each $\alpha \in \mathbb{C}$.
- The specialization of $\mathrm{Fitt}_i\,\mathscr{M}_t^*$, by $t \to \alpha$, is $\mathrm{Fitt}_i\,\mathscr{M}_\alpha^*$.

Lemma 3.6. *Let* $\alpha \in \mathbb{C}$. *Then* $\mathrm{syz}_1^{R[t]}\left(\mathscr{M}_t\right)/(t-\alpha)\,\mathrm{syz}_1^{R[t]}\left(\mathscr{M}_t\right) \cong \mathrm{syz}_1^R(\mathscr{M}_\alpha)$.

Proof. In $R[t]$ we have the following short exact sequence:

$$0 \longrightarrow \mathrm{syz}_1^{R[t]}\left(\mathscr{M}_t\right) \longrightarrow R[t]^6 \longrightarrow \mathscr{M}_t \longrightarrow 0.$$

Note that all of these are $\mathbb{C}[t]$-modules, and so tensoring this sequence by $\mathbb{C}[t]/(t-\alpha)$ over $\mathbb{C}[t]$, we have the long exact sequence:

$$\ldots \to \operatorname{Tor}_1^{\mathbb{C}[t]}\left(\mathscr{M}_t, \frac{\mathbb{C}[t]}{(t-\alpha)}\right) \to \frac{\operatorname{syz}_1^{R[t]}(\mathscr{M}_t)}{(t-\alpha)\operatorname{syz}_1^{R[t]}(\mathscr{M}_t)} \to R^6 \to \mathscr{M}_\alpha \to 0.$$

Thus, $\frac{\operatorname{syz}_1^{R[t]}(\mathscr{M}_t)}{(t-\alpha)\operatorname{syz}_1^{R[t]}(\mathscr{M}_t)} \cong \operatorname{syz}_1^R(\mathscr{M}_\alpha)$ if $\operatorname{Tor}_1^{\mathbb{C}[t]}\left(\mathscr{M}_t, \frac{\mathbb{C}[t]}{(t-\alpha)}\right) = 0$. We can compute this Tor by finding a resolution of $\mathbb{C}[t]/(t-\alpha)$ over $\mathbb{C}[t]$and tensoring this resolution with $\mathscr{M}_t$. A minimal resolution for $\mathbb{C}[t]/(t-\alpha)$ is given by the Koszul complex of $t-\alpha$ over $\mathbb{C}[t]$. However, as we saw in Section 3.1, $t-\alpha$ is regular on $\mathscr{M}_t$. Thus, the Koszul complex will remain exact after tensoring the complex with $\mathscr{M}_t$. Therefore, each homology module of the tensored complex will be zero; and since $\operatorname{Tor}_1^{\mathbb{C}[t]}\left(\mathscr{M}_t, \frac{\mathbb{C}[t]}{(t-\alpha)}\right)$ is equal to the first homology module, we have $\operatorname{Tor}_1^{\mathbb{C}[t]}\left(\mathscr{M}_t, \frac{\mathbb{C}[t]}{(t-\alpha)}\right) = 0$. □

Notice in the proof above there was nothing we used that was unique to the module $\mathscr{M}_t$, except that it was a submodule of a free module. Thus, in general, we have for any submodule of a free module, $t-\alpha$ will be regular on that module; and so the syzygies of such a module will be preserved under specialization. By this fact, we will get the proof of the third lemma as a corollary. The proof of the second lemma follows from Theorem 1.12 of [13], as we show below.

Lemma 3.7. *For each $\alpha \in \mathbb{C}$, $\mathscr{M}_t^*/(t-\alpha)\mathscr{M}_t^* \cong \mathscr{M}_\alpha^*$.*

Proof. Let $\alpha \in \mathbb{C}$, $\varphi : \mathscr{M}_\alpha^* \to \mathscr{M}_t^*/(t-\alpha)\mathscr{M}_t^*$ be the natural map we get from the specialization $\mathscr{M}_t \to \mathscr{M}_\alpha$, and $\mathfrak{m}$ be the maximal ideal of R. For $\mathfrak{p} \in \operatorname{Spec} R \backslash \{\mathfrak{m}\}$, we have R_p is a regular local ring of dimension two or less. Since $(\mathscr{M}_t^*/(t-\alpha)\mathscr{M}_t^*)_\mathfrak{p}$ and $(\mathscr{M}_\alpha^*)_\mathfrak{p}$ are reflexive, both are maximal Cohen-Macaulay modules over $R_\mathfrak{p}$, by Theorem 1.9 in [13]. However, over a regular local ring, maximal Cohen-Macaulay is equivalent to free, by the Auslander-Buchsbaum formula. Thus, over $R_\mathfrak{p}$, $(\mathscr{M}_t^*/(t-\alpha)\mathscr{M}_t^*)_\mathfrak{p}$ and $(\mathscr{M}_\alpha^*)_\mathfrak{p}$ are free modules. It is a routine linear algebra exercise to check $\varphi_\mathfrak{p}$ is an isomorphism. Thus, φ is an isomorphism on $\operatorname{Spec} R \backslash \{\mathfrak{m}\}$. Finally, φ extends to an isomorphism over all of Spec R, by Theorem 1.12 in [13]. Therefore, $\mathscr{M}_t^*/(t-\alpha)\mathscr{M}_t^* \cong \mathscr{M}_\alpha^*$. $\square$

Now with the second lemma, the third lemma follows as a corollary.

Corollary 3.8. *The specialization of* $\mathrm{Fitt}_i(\mathscr{M}_t^*)$*, by* $t \to \alpha$*, is the ideal* $\mathrm{Fitt}_i(\mathscr{M}_\alpha^*)$ *in* R*.*

Proof. First, note that $\mathscr{M}_t^* \cong \mathrm{syz}_1^{R[t]} \operatorname{im} A^T$, where A is a presentation matrix of $\mathscr{M}_t$. Hence, $\mathscr{M}_t^*$ is the submodule of a free module. Then, from the discussion above, we have that, under the specialization $t \to \alpha$, a presentation matrix for $\mathscr{M}_t^*$ will specialize to a presentation matrix for $\mathscr{M}_\alpha^*$. Furthermore, taking minors of a matrix commutes with specialization, and so the corollary follows. $\square$

In the next section, we have to deal with the question of whether a standard basis of an ideal will remain a standard basis after specialization. There are some results known on this topic, see [12]. We will deal with this question on a case by case basis.

For the computation of the Fitting ideals of $\mathscr{M}_t^*$ we will be using Singular. Singular has the capability to perform calculation over local orderings. This allows for computations over the ring $\mathbb{Q}[u,v,x,y,z]_{(u,v,x,y,z)}$, the localization of $\mathbb{Q}[u,v,x,y,z]$ at the maximal ideal. Recall in Section 2.2, we outlined why computations over $\mathbb{Q}[u,v,x,y,z]$ would be the same over $\mathbb{C}[u,v,x,y,z]$. This is also the case for the localizations of $\mathbb{Q}[u,v,x,y,z]$ and $\mathbb{C}[u,v,x,y,z]$ at the maximal ideal, by the same argument. Furthermore, since the standard basis algorithm on a set of polynomials always returns a set of polynomials and we have the following inclusion of rings, $\mathbb{C}[u,v,x,y,z]_{(u,v,x,y,z)} \subset \mathbb{C}[\![u,v,x,y,z]\!]$, any standard basis over $\mathbb{C}[u,v,x,y,z]_{(u,v,x,y,z)}$ will remain a standard basis over $\mathbb{C}[\![u,v,x,y,z]\!]$. Singular can keep a local ordering on the variables u,v,x,y,z, and give monomials in t a global ordering. Thus, with Singular we can perform computations over the rings $R[t]$ from our family $\mathscr{F}$. For further discussion on computations in rings of mixed orderings, see [12] and [5].

3.3 Fitting Ideals of $\mathscr{M}_\alpha$

We now turn to the computation of the needed Fitting ideals of $\mathscr{M}_t^*$ using Singular. Throughout this section, we denote the dual of $\mathscr{M}_t$ as $\mathscr{M}_t^* = \mathrm{Hom}_{R[t]}(\mathscr{M}_t, R[t])$. We will first look at the $E_6^\sharp$, $E_7^\sharp$, and $E_8^\sharp$ cases, because these require less computations. For these three cases, we will be considering the third Fitting ideal of $\mathscr{M}_t^*$.

Let R be the local ring of the $E_6^\sharp$ singularity. In order to compute the Fitting ideals of $\mathscr{M}_t^*$, we must first find $\mathscr{M}_t^*$. We can do this by computing the first syzygies of the image of the transpose of a presentation matrix of $\mathscr{M}_t$. Recall a presentation matrix for $\mathscr{M}_t$ is equivalent to a generating set for the syzygies of $\mathscr{M}_t$. Thus, to find a standard basis for $\mathscr{M}_t^*$ over $R[t]$, we can run the following lines of code in Singular.

```
ring S=0,(u,v,x,y,z,t),(ds(5),dp);
matrix a[3][2]=x,u,y,v,z,x3+y4;
ideal D = minor(a,2);
qring R = std(D);
module M=[x^2,0],[xy,0],[xz,0],[0,x2],[0,ux],[ux+tvx,u2];
matrix mM = M;
matrix msyzM=syz(M);
matrix rmsyzM=std(msyzM);
matrix msyzMT=transpose(msyzM);
module HMR = syz(msyzMT);
matrix mHMR=HMR;
matrix rHMR = std(HMR);
print(rHMR);
```

Figure 3.1 Singular code to produce a generating set for $\mathscr{M}_t^*$.

This code produces the following matrix:

$$\begin{bmatrix} x & -u & 0 & 0 & xz \\ y & -v & 0 & 0 & yz \\ z & -x^3-y^4 & 0 & 0 & z^2 \\ 0 & x+yt & z & y^2 & 0 \\ 0 & u+vt & x^3+y^4 & vy & 0 \\ u+vt & 0 & ux^2+vy^3 & v^2 & x^4+x^3yt+xy^4+y^5t \end{bmatrix}.$$

Thus, $\mathscr{M}_t^*$ is generated by the column space of this matrix. Now to compute the Fitting ideals for this module, we need the presentation matrix of $\mathscr{M}_t^*$. Using the Singular code below we can produce the presentation matrix for $\mathscr{M}_t^*$.

```
matrix sHMR=syz(rHMR);
print(sHMR);
```

Figure 3.2 Singular code to produce a presentation matrix for $\mathscr{M}_t^*$.

Thus, the presentation matrix for $\mathscr{M}_t^*$ is

$$\begin{bmatrix} -z & -v^2 & -ux^2-uy^2 & -uxy-vxyt+uy^2t-vy^2 & -vy & -x^3-xy^2 & 0 \\ 0 & -vy & -x^3-xy^2 & -x^2y-y^3 & -y^2 & -z & 0 \\ 0 & 0 & u+tv & v & 0 & x+yt & -y^2 \\ 0 & u+vt & 0 & x^2t+y^2t & x+yt & 0 & z \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By definition the third Fitting ideal is given by the $(n-3)$ by $(n-3)$ minors of a presentation matrix for $\mathscr{M}_t^*$, where n is the number of generators of $\mathscr{M}_t^*$. Recall the number of rows of a presentation matrix is always the number of generators for the module. We can produce a standard basis for the third Fitting ideal of $\mathscr{M}_t^*$, by running the following Singular code.

```
ideal fitt3=minor(sHMR,nrows(sHMR)-3);
ideal stdfitt3=std(fitt3);
stdfitt3;
```

Figure 3.3 Singular code to produce a standard basis for $\operatorname{Fitt}_3 \mathscr{M}_t^*$.

This will produce the standard basis $G = \{u+tv, v, x+yt, z, y^2\}$ for the third Fitting ideal of $\mathscr{M}_t^*$. Next, we turn to the question of how the third Fitting ideal and this standard basis will specialize under the map $t \to \alpha$. For $\alpha \in \mathbb{C}$ we have the specialization of $\operatorname{Fitt}_3 \mathscr{M}_t^*$ is the ideal $(u+\alpha v, v, x+\alpha y, z, y^2)$. From Section 3.2, we know this is equal to $\operatorname{Fitt}_3 \mathscr{M}_\alpha^*$. Thus, we have

$$\operatorname{Fitt}_3 \mathscr{M}_\alpha^* = (u + \alpha v, v, x + \alpha y, z, y^2) = (u, v, x + \alpha y, z, y^2).$$

We claim the set of generators for $\operatorname{Fitt}_3 \mathscr{M}_\alpha^*$ above form a standard basis. Let $G = \{u, v, x + \alpha y, z, y^2\}$. To show G is a standard basis recall in the computation of a standard basis we find all s-series (polynomials in our case) and find any remainders after expressing these in Mora's normal form with respect to our set G. However, the s-series between two monomials is always 0, since the leading term of a monomial is itself. Therefore, we can only get non-trivial remainders from our set G from an s-series involving $x + \alpha y$. Suppose $w \in G \backslash \{x + \alpha y\}$. Then, we have

$$s(w, x + \alpha y) = \frac{wx}{w} w - \frac{wx}{x}(x + \alpha y) = wx - wx - \alpha wy = \alpha y(w).$$

Since w is in G, we have that the remainder of $\alpha y(w)$ in Mora's normal form is 0. Therefore, the set G is closed under s-series, and so G is a standard basis.

Now suppose $\alpha, \beta \in \mathbb{C}$, such that $\alpha \neq \beta$. Then, we have $x + \alpha y \in \operatorname{Fitt}_3 \mathscr{M}_\alpha^*$; and claim $x + \alpha y \notin \operatorname{Fitt}_3 \mathscr{M}_\beta^*$. We reduce $x + \alpha y$ with respect to the standard basis $G_\beta = \{u, v, x + \beta y, z, y^2\}$:

$$x + \alpha y = x + \beta y + (\alpha - \beta) y.$$

Thus, the remainder, when expressing the normal form of $x + \alpha y$ with respect to the standard basis of $\operatorname{Fitt}_3 \mathscr{M}_\beta$, is $(\alpha - \beta) y \neq 0$, and so $x + \alpha y \notin \operatorname{Fitt}_3 \mathscr{M}_\beta^*$. Hence, $\operatorname{Fitt}_3 \mathscr{M}_\alpha \neq \operatorname{Fitt}_3 \mathscr{M}_\beta$. Therefore, $\mathscr{M}_\alpha^*$ is not isomorphic to $\mathscr{M}_\beta^*$. This in turn implies that $\mathscr{M}_\alpha$ is not isomorphic to $\mathscr{M}_\beta$. Thus, for each $\alpha \in \mathbb{C}$, we get a distinct module $\mathscr{M}_\alpha$. Joining this result with the work from earlier in this chapter, we have proven the following:

Theorem 3.9. *Let R be the type $E_6^\sharp$ local ring from our family, $\mathscr{F}$. Then, for each*

$\alpha \in \mathbb{C}$*, the* R*-module,* $\mathscr{M}_\alpha$*, generated by the column space of the matrix,*

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & xu + \alpha xv \\ 0 & 0 & 0 & x^2 & xu & u^2 \end{bmatrix},$$

is a rank two maximal Cohen-Macaulay module. Moreover, if $\alpha \neq \beta \in \mathbb{C}$*, then* $\mathscr{M}_\alpha \ncong \mathscr{M}_\beta$*.*

We claim that this implies that R has an infinite set of indecomposable maximal Cohen-Macaulay modules. If $\mathscr{M}_\alpha$ is indecomposable for any $\alpha \in \mathbb{C}$, say $\mathscr{M}_\alpha \cong I_\alpha \oplus J_\alpha$, then I_α and J_α are rank one maximal Cohen-Macaulay modules. Suppose α, β are such that $\alpha \neq \beta$ and $\mathscr{M}_\alpha \cong I_\alpha \oplus J_\alpha$ and $\mathscr{M}_\beta \cong I_\beta \oplus J_\beta$. Then $\mathscr{M}_\alpha \ncong \mathscr{M}_\beta$, implies that $I_\alpha \ncong I_\beta$, $I_\alpha \ncong J_\beta$, $J_\alpha \ncong I_\beta$, or $J_\alpha \ncong J_\beta$. Thus, if $\mathscr{M}_\alpha$ is decomposable, at least one of the summands, I_α or J_α, must be distinct. Therefore, R has an infinite set of rank one maximal Cohen-Macaulay modules or an infinite set of rank two maximal Cohen-Macaulay modules. Consequently, we have proven the main theorem for the type $E_6^\sharp$ case.

Theorem 3.10. *The local ring of a type* $E_6^\sharp$ *singularity has infinite Cohen-Macaulay representation type.*

Using similar Singular code, we can compute a standard basis for the third Fitting ideal of $\mathscr{M}_t^*$ over the rings $R[t]$, where R is the local ring of type $E_7^\sharp$ or $E_8^\sharp$. In fact, for these two cases we get that the standard basis for $\operatorname{Fitt}_3 \mathscr{M}_t^*$ is the same as the $E_6^\sharp$ case. Thus, as in the $E_6^\sharp$ case, we will have the following theorems, since the discussion above will hold over these two rings as well.

Theorem 3.11. *Let* R *be the type* $E_7^\sharp$ *or* $E_8^\sharp$ *local ring from our family,* $\mathscr{F}$*. Then, for each* $\alpha \in \mathbb{C}$*, the* R*-module,* $\mathscr{M}_\alpha$*, generated by the column space of the matrix,*

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & xu + \alpha xv \\ 0 & 0 & 0 & x^2 & xu & u^2 \end{bmatrix},$$

is a maximal Cohen-Macaulay module. Moreover, if $\alpha \neq \beta \in \mathbb{C}$*, then* $\mathscr{M}_\alpha \ncong \mathscr{M}_\beta$*.*

Theorem 3.12. *The local ring of a type $E_7^\sharp$ or type $E_8^\sharp$ singularity has infinite Cohen-Macaulay representation type.*

Now we consider the $D_k^\sharp$ case. In principle, we are doing the same thing as in the $E_6^\sharp$ case, except for varying values of k. First, the procedure below can be implemented in Singular to quickly get the standard basis of the third Fitting ideal of $\mathscr{M}_t^*$ for varying k.

This procedure takes in an integer, k, for $k \geq 4$, and prints a standard basis for the third Fitting ideal of $\mathscr{M}_t$ over $R[t]$, where R is the local ring of the $D_k^\sharp$ singularity.

```
proc getFitt3(int k)
{
matrix a[3][2]=x,u,y,v,z,x^(k-1)+xy2;
ideal D = minor(a,2);
qring R = std(D);
module M=[x^2,0],[xy,0],[xz,0],[0,x2],[0,ux],[ux+tvx,u2];
matrix msyzM=syz(M);
matrix rmsyzM=std(msyzM);
matrix msyzMT=transpose(msyzM);
matrix HMR = syz(msyzMT);
matrix rHMR = std(HMR);
matrix sHMR=syz(rHMR);
ideal fitt3=minor(sHMR,2);
ideal stdfitt3=std(fitt3);
stdfitt3;
}
```

Figure 3.4 Singular code to produce $\mathrm{Fitt}_3\,\mathscr{M}_t^*$.

Running this for varying values of k the procedure returns the standard basis $G = \{u + vt, v, x + yt, z, y^2\}$ for each k. To check its validity, for $4 \leq k \leq 2000$, we use the Singular code below.

This procedure takes in a range of values for k starting at s and stopping at f, computes a standard basis for $\mathrm{Fitt}_3\,\mathscr{M}_t^*$ over the ring $R[t]$, compares this standard basis to the claimed basis $G = \{u + vt, v, x + yt, z, y^2\}$, and then stores either 1 for true or 0 for false for each value of k in a list. The procedure then returns this list of 0's and 1's.

```
proc checkFitt(int s,int f)
{
list l;
    for(int k=s;k<=f;k=k+1)
    {
     setring S;
     matrix a[3][2]=x,u,y,v,z,x^(k-1)+xy^2;
     ideal D = minor(a,2);
     qring R = std(D);
     module M=[x^2,0],[xy,0],[xz,0],[0,x2],[0,ux],[ux+tvx,u2];
     matrix msyzM=syz(M);
     matrix rmsyzM=std(msyzM);
     matrix msyzMT=transpose(msyzM);
     matrix HMR = syz(msyzMT);
     matrix rHMR = std(HMR);
     matrix sHMR=syz(rHMR);
     ideal fitt3=minor(sHMR,2);
     ideal stdf3=std(fitt3);
     vector Vstdf3=[stdf3[1],stdf3[2],stdf3[3],stdf3[4],stdf3[5]];
     vector Vtest=[u+vt,v,x+yt,z,y2];
     l[k]=(Vstdf3-Vtest==0);
    }
return(l);
}
```

Figure 3.5 Singular code to check standard basis over a range.

Finally, we can make a simple procedure to check if every value in a list is equal 1:

```
proc checkTrue(list L, int s, int f)
{
    int switch=1;
    for(int k=s;k<=f;k=k+1)
    {
        if(L[k]==0)
        {
            switch=0;
        }
    }
return(switch);
}
```

Figure 3.6 Singular code to check if a list has 1 as every entry.

The input of this procedure is the starting index, s, of a list and the last index, f, of the list and returns 1 for true if every entry in the list is 1 or 0 if any entry in the list is not equal to 1. Using the checkFitt and checkTrue procedures for the range of k from 4 to 2000, we have the following theorem:

Theorem 3.13. *For $4 \leq k \leq 2000$, $G = \{u + vt, v, x + yt, z, y^2\}$ is a standard basis for $\operatorname{Fitt}_3 \mathscr{M}_t^*$ over the local ring of the $D_k^\sharp$ singularity.*

Now, just as in the $E_k^\sharp$ cases, specializing G by $t \to \alpha$ will give us that $\operatorname{Fitt}_3 \mathscr{M}_\alpha^* = (u, v, x + \alpha y, z, y^2)$. Furthermore, just as in the $E_k^\sharp$ cases, we have this is, in fact, a standard basis for $\operatorname{Fitt}_3 \mathscr{M}_\alpha^*$, by the same argument. Therefore, we see that each $\mathscr{M}_\alpha$ is distinct, since $\operatorname{Fitt}_3 \mathscr{M}_\alpha^* \neq \operatorname{Fitt}_3 \mathscr{M}_\beta^*$. Thus, we have the following theorem:

Theorem 3.14. *Let R be the type $D_k^\sharp$ local ring from our family, $\mathscr{F}$ for $4 \leq k \leq 2000$. Then, for each $\alpha \in \mathbb{C}$, the R-module, $\mathscr{M}_\alpha$, generated by the column space of the matrix,*

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & xu + \alpha xv \\ 0 & 0 & 0 & x^2 & xu & u^2 \end{bmatrix},$$

is a rank two maximal Cohen-Macaulay module. Moreover, if $\alpha \neq \beta \in \mathbb{C}$, then $\mathscr{M}_\alpha \ncong \mathscr{M}_\beta$.

As in the $E_k^\sharp$ cases, this implies, for each R of type $D_k^\sharp$, with $4 \leq k \leq 2000$, that R has an infinite family of indecomposable maximal Cohen-Macaulay modules. Therefore, we have proven the main theorem in the $D_k^\sharp$ case, for $4 \leq k \leq 2000$.

Theorem 3.15. *The local ring of a type $D_k^\sharp$ singularity has infinite Cohen-Macaulay representation type for $4 \leq k \leq 2000$.*

Lastly, we need to consider the local rings of the type $A_k^\sharp$ singularities. We can use similar Singular code to find the third Fitting ideal of $\mathscr{M}_t^*$. For the ring $R[t]$, where R is the local ring of the $A_1^\sharp$ singularity, Singular returns the standard basis, $G = \{u + vt, v, x + yt, yt^2 + y, z, y^2\}$. Specializing this by $t \to \alpha$, we have the third Fitting ideal for $\mathscr{M}_\alpha^*$ over the $A_1^\sharp$ singularity is $(u + \alpha v, v, x + \alpha y, \alpha^2 y + y, z, y^2) = (u, v, x, y, z)$. Thus, this will not be helpful in showing $\mathscr{M}_\alpha^* \ncong \mathscr{M}_\beta^*$. Checking for other values of k yields similar results. Consequently, we will instead compute the second Fitting ideal of $\mathscr{M}_t^*$.

We consider the second Fitting ideal for $\mathscr{M}_t^*$ over the $A_1^\sharp$ separately, since the standard basis in this case is slightly different. Thus, suppose R is the local ring of the $A_k^\sharp$ singularity, for $2 \leq k \leq 2000$, then a standard basis for $\operatorname{Fitt}_2 \mathscr{M}_t^*$ over $R[t]$ is

$$G = \{u^2 + uvt, uv + v^2t, ux + vxt + uyt + vyt^2, x^2 + 2txy + y^2t^2, uy + vyt, xy + y^2t,$$
$$xz, yz, z^2, v^3, v^2y, vy^2, y^3\}.$$

The following Singular procedure can be used to show this.

```
proc checkFitt2(int k)
{
matrix a[3][2]=x,u,y,v,z,x^(k+1)+y2;
ideal D = minor(a,2);
qring R = std(D);
module M=[x^2,0],[xy,0],[xz,0],[0,x2],[0,ux],[ux+tvx,u2];
matrix msyzM=syz(M);
matrix rmsyzM=std(msyzM);
matrix msyzMT=transpose(msyzM);
matrix HMR = syz(msyzMT);
matrix rHMR = std(HMR);
matrix sHMR=syz(rHMR);
ideal fitt2=minor(sHMR,nrows(sHMR)-2);
ideal stdfitt2=std(fitt2);
ideal G=u2+uvt,uv+v2t,ux+vxt+uyt+vyt2,x2+2xyt+y2t2,uy+vyt,
xy+y2t,xz,yz,z2,v3,v2y,vy2,y3;
ideal stdG=std(G);
ideal L1=reduce(stdfitt2,stdG);
ideal L2=reduce(stdG,stdfitt2);
vector V1=[L1[1],L1[2],L1[3],L1[4],L1[5],L1[6],L1[7],L1[8],
L1[9],L1[10],L1[11],L1[12],L1[13]];
vector V2=[L2[1],L2[2],L2[3],L2[4],L2[5],L2[6],L2[7],L2[8],
L2[9],L2[10],L2[11],L2[12],L2[13]];
vector Z=[0,0,0,0,0,0,0,0,0,0,0,0,0];
return((V1==Z) and (V2==Z));
}
```

Figure 3.7 Singular code to check G is a standard basis.

The procedure takes in an integer $k \geq 2$ and computes a standard basis, G', for $\operatorname{Fitt}_2 \mathscr{M}_t^*$ over $R[t]$. Then the procedure reduces G' with respect to the claimed standard basis, G, storing the result of this reduction in a vector. Next, the procedure reduces G with respect to G', storing the result of this reduction in a vector as well. If all entries in these vectors are zero, then we have G and G' generate each other. The procedure then compares both vectors to the 0 vector and returns 1 if all entries in both are zero, or 0 if any entry in either vector is non-zero. Looping this procedure, over the range $k = 2$ to $k = 2000$, proves the claimed G is a standard basis for $\operatorname{Fitt}_2 \mathscr{M}_t^*$ over $R[t]$.

Hence, specializing by $t \to \alpha$ we have

$$\operatorname{Fitt}_2 \mathscr{M}_\alpha^* = (u^2 + \alpha uv, uv + \alpha v^2, ux + \alpha vx + \alpha uy + \alpha^2 vy,$$
$$x^2 + 2\alpha xy + \alpha^2 y^2, uy + \alpha vy, xy + \alpha y^2, xz, yz, z^2, v^3, v^2 y, vy^2, y^3)$$

over the $A_k^\sharp$ singularity for $2 \le k \le 2000$. Finally, we claim that the set of generators for $\operatorname{Fitt}_2 \mathscr{M}_\alpha^*$ above form a standard basis. By the modified Buchberger criteria for a standard basis, this is a standard basis if it is closed under taking s-series (again in our case polynomials). Recall from the $E_6^\sharp$ case, a non-zero remainder from an s-series can only arise from the s-series of two non-monomials. Hence, labeling the first six generators of $\operatorname{Fitt}_2 \mathscr{M}_\alpha$, $g_1, \ldots, g_6$, respectively, we only need to check the remainders of $s(g_i, g_j)$ for $1 \le j < i \le 6$. Keeping in mind over each R we have the relation $vx = uy$, this is an easy calculation, which we leave to the reader. Now we show for $\alpha, \beta \in \mathbb{C}$, such that $\alpha \neq \beta$, $\operatorname{Fitt}_2 \mathscr{M}_\alpha^* \neq \operatorname{Fitt}_2 \mathscr{M}_\beta^*$. We claim $u^2 + \beta uv \notin \operatorname{Fitt}_2 \mathscr{M}_\alpha^*$. Reducing $u^2 + \beta uv$, with respect to the standard basis for $\mathscr{M}_\alpha^*$, we have

$$u^2 + \beta uv = u^2 + \alpha uv - (\alpha - \beta)(uv + \alpha v^2) + (\beta - \alpha) v^2.$$

Due to the fact that in negative graded reverse lexicographical order v^2 is less than u^2, and no leading term in the standard basis for $\operatorname{Fitt}_2 \mathscr{M}_\alpha^*$ divides v^2, we have a non-zero remainder from reducing $u^2 + \beta uv$. Hence, $u^2 + \beta uv \notin \operatorname{Fitt}_2 \mathscr{M}_\alpha^*$. Therefore, the two Fitting ideals are not equal. As before, this implies $\mathscr{M}_\alpha \ncong \mathscr{M}_\beta$.

Now let R be the local ring of the $A_1^\sharp$ singularity. With the help of Singular, we have that a standard basis for $\operatorname{Fitt}_2 \mathscr{M}_t^*$ over $R[t]$ is $G = \{u^2 + 2uvt + v^2t^2, uv + v^2t, ux + vxt + uyt + vyt^2, x^2 + 2xyt + y^2t^2, uy + vyt, xyt^2 + xy + y^2t^3 + y^2t, xz + yzt, yzt^2 + yz, z^2, v^3, v^2y, vy^2, xy^2 + y^3t, y^3t^2 + y^3, y^2z, y^4\}$.

Thus specializing, by $t \to \alpha$ we have

$$\operatorname{Fitt}_2 \mathscr{M}_\alpha = (u^2 + 2\alpha uv + \alpha^2 v^2, uv + \alpha v^2, ux + \alpha vx + \alpha uy + \alpha^2 vy, x^2 + 2\alpha xy + \alpha^2 y^2,$$
$$uy + \alpha vy, (1+\alpha^2)xy + \alpha(1+\alpha^2)y^2, xz + \alpha yz, (1+\alpha^2)yz,$$
$$z^2, v^3, v^2y, vy^2, xy^2 + \alpha y^3, (1+\alpha^2)y^3, y^2z, y^4)$$

over the local ring of the $A_1^\sharp$ singularity. The factors of $1+\alpha^2$ appearing in the ideal will yield a different standard basis when $\alpha = i$, and so we will deal with this case first. For $\alpha = i = \sqrt{-1}$, a standard basis for $\operatorname{Fitt}_2 \mathscr{M}_i^*$ is

$$G_i = \{u^2 + 2iuv - v^2, uv - v^2, ux + ivx + iuy - vy, x^2 + 2ixy - y^2$$
$$uy + ivy, xz + iyz, z^2, v^3, v^2y, yv^2, xy^2 + iy^3, y^2z, y^4\}.$$

Now assume $\alpha \neq i$. First, we remove some of the redundant generators, and then show the resulting generating set is a standard basis. Denote the current generators of $\operatorname{Fitt}_2 \mathscr{M}_\alpha^*$, by $g_1, \ldots, g_{16}$, respectively. We claim that for all other α a standard basis for $\operatorname{Fitt}_2 \mathscr{M}_\alpha$ is

$$\{u^2 + \alpha uv, uv + \alpha v^2, ux + \alpha vx + \alpha uy + \alpha^2 vy, x^2 + 2\alpha xy + \alpha^2 y^2,$$
$$uy + \alpha vy, xy + \alpha y^2, xz + \alpha yz, yz, z^2, v^3, v^2y, vy^2, y^3\}.$$

Denote these generators by $f_1, \ldots, f_{13}$. Clearly, we have $\operatorname{Fitt}_2 \mathscr{M}_\alpha^* = (f_1, \ldots, f_{13})$, since $\alpha^2 + 1$ is a unit. To show this is a standard basis, notice the f_i's are the same generators of the standard basis for $\operatorname{Fitt}_2 \mathscr{M}_\alpha^*$ in the $A_k^\sharp$ case for $2 \leq k \leq 2000$. Hence, the computation of the s-series will be the same for this case. Therefore, $G = \{f_1, \ldots, f_{13}\}$ is a standard basis for $\operatorname{Fitt}_2 \mathscr{M}_\alpha^*$, for $\alpha \neq i$.

Now let $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq \beta$. We claim $\operatorname{Fitt}_2 \mathscr{M}_\alpha^* \neq \operatorname{Fitt}_2 \mathscr{M}_\beta^*$. To see this, we have $uy + \alpha vy$ is in a standard basis for $\operatorname{Fitt}_2 \mathscr{M}_\alpha^*$. Reducing this, with respect to the standard basis we found for $\operatorname{Fitt}_2 \mathscr{M}_\beta^*$, we have

$$uy + \alpha vy = uy + \beta vy + (\alpha - \beta)vy.$$

In our monomial ordering we have $vy < uy$ and no other leading terms in our standard basis for $\operatorname{Fitt}_2 \mathscr{M}_\beta$ divide vy. Thus, the remainder of reducing $uy+\alpha vy$ is $(\alpha-\beta)vy \neq 0$. Therefore, $uy + \alpha vy \notin \operatorname{Fitt}_2 \mathscr{M}_\beta^*$, and so $\operatorname{Fitt}_2 \mathscr{M}_\alpha^* \neq \operatorname{Fitt}_2 \mathscr{M}_\beta^*$. Hence, as in the previous cases, for each $\alpha \in \mathbb{C}$, $\mathscr{M}_\alpha$ is distinct. The discussion on indecomposability, from the $E_k^\sharp$ and $D_k^\sharp$ cases, holds for this case as well. Consequently, we get the two theorems below in the $A_k^\sharp$ case.

Theorem 3.16. *Let R be the type $A_k^\sharp$ local ring from our family, $\mathscr{F}$ for $1 \leq k \leq 2000$. Then, for each $\alpha \in \mathbb{C}$, the R-module, $\mathscr{M}_\alpha$, generated by the column space of the matrix,*

$$\begin{bmatrix} x^2 & xy & xz & 0 & 0 & xu + \alpha xv \\ 0 & 0 & 0 & x^2 & xu & u^2 \end{bmatrix},$$

is a rank two maximal Cohen-Macaulay module. Moreover, if $\alpha \neq \beta \in \mathbb{C}$, then $\mathscr{M}_\alpha \ncong \mathscr{M}_\beta$.

Theorem 3.17. *The local ring of a type $A_k^\sharp$ singularity has infinite Cohen-Macaulay representation type for $1 \leq k \leq 2000$.*

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