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## Deducing Vertex Weights From Empirical Occupation Times

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DEDUCING VERTEX WEIGHTS FROM EMPIRICAL OCCUPATION TIMES

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## DEDICATION

This is dedicated to my wonderful wife Britney; I couldn't have done it without you. You are my pigeon, I will always come home to you.

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There are so many people that have helped me get where I am today. I would first like to thank my advisor, for introducing me to the problem, encouraging my progress, and guiding my random walk towards a dissertation.

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## ABSTRACT

Consider the following problem arising from the study of human problem solving: Let  $G$  be a vertex-weighted digraph with marked “start” and “end” vertices. Suppose that a random walker begins at the start vertex, steps to neighbors of vertices with probability proportional to their weights, and stops upon reaching the end vertex. Could one deduce the weights from the paths that many such walkers take? An iterative numerical solution to this reconstruction problem is analyzed for when the empirical mean occupation times of the walkers is given. The existence of a choice of weights that gives rise to a given list of expected occupation times is considered, showing several equivalent conditions for such a solution to exist, and giving an algorithm for finding a solution when there is one.

A generalization of projective space, which we refer to as "graphical projective space," arising from these questions is then considered which takes as an input a hypergraph. Some of the properties of these spaces are discussed, using a natural CW-complex to distinguish between them, and some small examples are given.

Finally, graphical projective spaces are applied as natural spaces for vertex weights on a graph, and the problem of how to extend the solution of the random-walk problem on the graph to the appropriate graphical projective space is considered. Several open problems are discussed.

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# CHAPTER 1

## INTRODUCTION

Single-agent search problems are commonly modeled as a graph  $G$ , with an edge from  $x \in V(G)$  to  $y \in V(G)$ , written  $x \rightarrow y$ , if it is possible to move from state  $x$  to state  $y$ . We will not make any assumptions about such moves being reversible, so that  $G$  may be a directed graph. We distinguish a vertex  $v_{\text{start}}$  as the starting vertex, and  $v_{\text{end}}$  as an ending vertex. These model the starting point and solution of a problem being solved.

Some typical examples of such single-agent search problems include:

1. Vertices are states of a Rubik's Cube or a 15-puzzle, with an edge between two vertices if it is possible to transform one into the other by an allowable move. Here  $v_{\text{start}}$  is the starting state, and  $v_{\text{end}}$  is the solved puzzle.
2. Vertices are web pages, with edges corresponding to hyperlinks. In this case,  $v_{\text{start}}$  may be a company homepage, and  $v_{\text{end}}$  a page where purchases are made.
3. Vertices are the positions of a chess board, edges correspond to legal moves,  $v_{\text{start}}$  is the initial position given by a chess puzzle, and  $v_{\text{end}}$  is the set of winning configurations.
4. Vertices are a grid of points in a mouse maze, with edges corresponding to feasible moves,  $v_{\text{start}}$  is the cage door and  $v_{\text{end}}$  is the cheese.

In many such examples, a researcher has access to the state of the solver, but not to their reasoning process (their “policy” to use machine learning parlance). The

amount of time a subject takes to find the solution state (the “latency”) can serve as a useful proxy for their knowledge level, but this single number is a somewhat crude measurement. One might strive to learn in addition the worth attributed by the solver to intermediate states, i.e. the solver’s “value function.” Such detailed profiles of preferences could aid in, for example, improving customer service, evaluating individual expertise, estimating how well a lab animal has learned a task, tuning a software game-playing engine, or identifying gaps in students’ knowledge. However, the solver, be it human, lab animal, or machine, may be long gone, may not have conscious knowledge of this information, may be secretive, or may not be able to express their thoughts in a human-readable format. Nonetheless, by studying the path that many instances of the solver take, one could hope to reconstruct such valuational ascriptions without the involvement of the solvers. This strategy is akin to using the density of oil stains in a parking lot to see which spots are most popular, or evaluating historical road use by the depth of wheel-ruts [26].

This paper models the solution process as a random walk on the graph  $G$ , starting at  $v_{\text{start}}$  and ending at  $v_{\text{end}}$ . A novice solver may follow something as simple as a uniform random walk, with each neighbor at a given step having the same probability as any other neighbor. A more experienced solver might follow a more direct route through the graph, as they will be inclined to move closer to the solution state with each move.

Much study of random walks has been done on simple graphs [15], occasionally using edge weights to modify the probabilities. More recently, random walks on directed graphs with edge weights have been studied to analyze the PageRank algorithm [3]. Vertex weights specify the proportional probabilities of moves, and encode the intuition of valuing a particular state, no matter where the path is coming from. Furthermore, the dimension of the space of vertex weights is the same as the dimension of the space of vectors of expected occupation times. This severely limits the

possible sets of weights which give rise to a particular vector of expected occupation times, whereas the space of edge weights which give rise to the same vector would be a multi-dimensional space.

In the second chapter, this model is described in greater detail and vertex weights are related to empirical mean occupation times. The problem of determining the weights from the occupation times is resolved by using an iterative algorithm to obtain a numerical solution. The analysis includes a characterization of solution existence: a description of which sets of occupation times admit a set of weights that give rise to it. This is followed by an analysis of the number of steps required, and the rate of convergence of the algorithms involved. Further results are proven for the special case of non-directed graphs.

Our main results are a formula to find the expected occupation times based on a set of weights, and a characterization of solution existence for the occupation times. The following two theorems give the results; all of the necessary notation is defined in Chapter 2.

**Proposition 1.** *The expected number of visits in a weighted random walk from the start vertex to the end vertex is given by the coordinates of the unique eigenvector  $D$  with  $D_n = 1$  associated with the eigenvalue 1 of the matrix*

$$\tilde{P} = \text{diag}(W) \cdot \tilde{A}^T \cdot (\text{diag}(\tilde{A} \cdot W))^{-1}$$

*That is,  $\tilde{P}D = D$ .*

**Theorem 3.** *The following are equivalent*

1. *There exists a set of weights  $W$  such that  $D$  is the vector of expected occupation times on  $G$ .*
2.  *$D$  is in the relative interior of  $\mathcal{C}$  with respect to  $\mathcal{H}$ .*

3. There exists  $\Omega \subseteq \Lambda$  which covers all edges of  $G$  such that  $D$  is a strict convex combination of the traces of walks in  $\Omega$ .

4. For every  $1 \leq i \leq c$  and for every  $S \subseteq C_i$

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S$$

with equality iff  $S = C_i$  or  $S = \emptyset$ .

5. For every  $S \subseteq V$ ,

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S$$

with equality iff  $S = \cup_{i \in \sigma} C_i$  for some  $\sigma \subseteq [c]$

6. For every set of representatives,  $R$ , there exists a set of weights  $W$  such that  $F_i(W) \leq 0$  for all  $i \in V - R$ .

In the third chapter an extension of the solution space is presented which will allow non-finite weights. This is discussed in a general format, where an analog of projective space is defined, examples of which can be parameterized via hypergraphs. The term “graphical projective space” will be used to refer to these spaces. The construction of these spaces are discussed, and some of the properties are described. This is followed by further analysis of the special case where the underlying hypergraph is a simple graph.

The main results from the third chapter are in the following theorems, with the necessary notation defined in that chapter.

**Theorem 46.**  $P_H$  forms a CW-complex with open cells  $\mathcal{E}_H$ .

**Theorem 47.** There is a bijection between the open cells  $C \in \mathcal{E}_H$  and the chains of non-empty subsets of the vertices,  $S, V = S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots \supseteq S_k = \emptyset$ , such that for  $0 \leq i < k$ , every vertex in  $S_i - S_{i+1}$  is connected to a vertex in  $S_{i-1} - S_i$  in the graph  $H - S_{i+1}$ .  $\bar{C}$  is isomorphic in the category of CW-complexes to  $P_{H \setminus S}$ .

**Corollary 51.** *The CW-complex  $P_H$  is isomorphic in the category of CW-complexes with natural CW-complex of the zonotope formed under projection onto the cut space  $\mathcal{B}$ .*

The fourth chapter combines graphical projective spaces with the vertex-weighted random walk problem discussed in chapter two; allowing the vertex weights to come from graphical projective spaces as opposed to being restricted to finite positive weights. Analogous results are proven to those in Chapter 1, primarily in the form of the following theorem:

**Theorem 4.** *The following are equivalent*

1. *There exists a set of weights  $W \in P_H$  such that  $I - \bar{P}(W)$  is invertible, and  $D \in \mathbb{R}_{>0}^n$  is the vector of expected occupation times on  $G$ .*
2.  *$D$  is in the intersection of  $\mathbb{R}_{>0}^n$  and  $\mathcal{C}$*
3. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path such that there exists  $\Omega \subseteq \Lambda$  which covers all edges of  $G'$  such that  $D$  is a strict convex combination of the traces of walks in  $\Omega$ .*
4. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path, with corresponding graph  $H'$  with  $c'$  components  $C'_i$  such that for  $1 \leq i \leq c'$  and for every  $S \subseteq C'_i$*

$$\sum_{v_j \in S} D_j - \xi_S \leq \sum_{v_j \in N^-(S)} D_j,$$

*with equality iff  $S = C'_i$  or  $S = \emptyset$ .*

5. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path, with corresponding graph  $H'$  with  $c'$  components  $C'_i$  such that for every  $S \subseteq V$ ,*

$$\sum_{v_j \in S} D_j - \xi_S \leq \sum_{v_j \in N^-(S)} D_j,$$

*with equality iff  $S = \cup_{i \in \sigma} C'_i$  for some  $\sigma \subseteq [c]$*

6. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path, with corresponding hypergraph  $H'$  such that for every set of representatives of  $H'$ ,  $R$ , there exists a set of weights  $W$  such that  $F_i(W) < 0$  for all  $i \in V - R$ .*
7. *There exists a set of weights  $W \in \mathbb{R}_{>0}^n$  such that  $D$  is the expected occupation times on  $G'$ .*

The conclusion discusses some of the remaining open problems that arise in connection with these questions.

## CHAPTER 2

### RANDOM WALKS ON VERTEX-WEIGHTED GRAPHS

#### 2.1 INTRODUCTION AND DEFINITIONS

The model used involves a directed graph  $G$  with vertex set  $V$ ,  $|V| = n$ , and two distinguished vertices  $v_{\text{start}}$  and  $v_{\text{end}}$ . The vertices are all labeled  $v_1, v_2, \dots, v_n$ , and without loss of generality it is assumed that  $v_{\text{start}} = v_1$  and  $v_{\text{end}} = v_n$ . Based on this vertex ordering, the adjacency matrix for  $G$ ,  $A$ , can be defined:  $A_{ij} = 1$  if  $v_i \rightarrow v_j$ . It is assumed that every vertex in  $G$  is on some walk from  $v_{\text{start}}$  to  $v_{\text{end}}$ , and similarly for every edge. If the graph which arises in some application does not have this property, then it makes sense to restrict to the largest subgraph which does, as these are the only edges and vertices which will participate in the relevant random walks. The weights of each vertex are given by  $W : V \rightarrow \mathbb{R}_{>0}$ . Here  $\mathbb{R}$  is the set of all real numbers, and  $\mathbb{R}_{>0}$  is the set of positive real numbers.  $W$  is treated as a vector in  $\mathbb{R}_{>0}^n$ , the cartesian product of  $n$  copies of  $\mathbb{R}_{>0}$ . To facilitate this  $W_i$  is used as an abbreviation for  $W(v_i)$ . For  $S \subseteq V$ , in-neighborhood of  $S$  is defined as  $N^-(S) = \{v \in V : \exists s \in S, v \rightarrow s\}$ , and the out-neighborhood of  $S$  as  $N^+(S) = \{v \in V : \exists s \in S, s \rightarrow v\}$ . Note that  $S \cap N^-(S)$  and  $S \cap N^+(S)$  are allowed to be non-empty. The expected occupation times of the vertices are given by  $D : V \rightarrow \mathbb{R}$ .  $D$  is treated as a vector as well, with  $D_i = D(v_i)$ . Note in particular that  $D(v_{\text{end}}) = D_n = 1$ , since all of the walks visit the end vertex exactly once.

The formulas involved in the calculations can be written concisely in matrix format, so some matrix notation is used. In particular,  $\text{diag}(X)$  is the square matrix

with the vector  $X$  along the diagonal and 0's everywhere else.  $I$  is the identity matrix, with the size being evident from context.  $\chi_i$  is the vector with a 1 in the  $i$ th coordinate and 0's everywhere else.  $|\cdot|$  is used to refer to the  $L^2$  norm.

Given  $x, y \in V$ ,  $P(x, y)$  is defined as the probability of moving from  $x$  to  $y$  during the random walk. If  $x \not\rightarrow y$ , then  $P(x, y) = 0$ . If  $x \rightarrow y$ ,

$$P(x, y) = \frac{W(y)}{\sum_{z \in N^+(x)} W(z)}.$$

Note that  $P$  can also be written in matrix form as

$$P = \text{diag}(W) \cdot A^T \cdot (\text{diag}(A \cdot W))^{-1},$$

where  $P_{ij} = P(v_j, v_i)$ , the probability of moving from  $v_j$  to  $v_i$ . Note that the roles of  $i$  and  $j$  are “reversed” from the convention employed for the adjacency matrix.

## 2.2 FINDING THE OCCUPATION TIMES

The first problem is to figure out how to find the expected occupation times given a particular graph and set of weights. First note that for any vertex except  $v_1$ ,

$$D_i = \sum_{j=1}^n P_{ij} D_j.$$

If this were true for all of the vertices, then  $D$  would be an eigenvector of  $P$ . Note that,  $N^+(v_n) = \emptyset$ , so the last column of  $A$  is all 0's. Define  $\tilde{A}$ , such that  $\tilde{A}_{ij} = A_{ij}$  if  $i \neq 1$  and  $j \neq n$ , and  $\tilde{A}_{1n} = 1$ . In essence this is the same as adding an edge from  $v_{\text{end}}$  to  $v_{\text{start}}$ . Using this definition the following proposition can be proven.

**Proposition 1.** *The expected number of visits in a weighted random walk from the start vertex to the end vertex is given by the unique eigenvector  $D$  with  $D_n = 1$  associated with the eigenvalue 1 of the matrix*

$$\tilde{P} = \text{diag}(W) \cdot \tilde{A}^T \cdot (\text{diag}(\tilde{A} \cdot W))^{-1}$$

*That is,  $\tilde{P}D = D$ .*

*Proof.* By the definition of  $\tilde{A}$ ,  $\tilde{P}_{ij} = P_{ij}$  except when  $i = 1$  and  $j = n$ , and  $\tilde{P}$  is still a probability matrix for some graph, call it  $\tilde{G}$ . Note that  $\tilde{G}$  is  $G$  with an added edge from  $v_n$  to  $v_1$ . As such, the columns of  $\tilde{P}$  sum to 1, so it is stochastic. By definition every vertex has a path in  $G$  to  $v_n$ , and a path from  $v_1$ , making  $\tilde{G}$  strongly connected.  $\tilde{A}$  is irreducible, and therefore  $\tilde{P}$  is as well. By the Perron-Frobenius theorem,  $\tilde{P}$  has 1 as an eigenvalue with a unique eigenvector  $E$  with  $E_n = 1$ , and all of its entries are positive. It remains to show that this vector  $E$  is the same as  $D$ . For this, recall that for  $i \neq 1$ ,

$$D_i = \sum_{j=1}^n P_{ij} D_j,$$

which can be rewritten as

$$D_i = \sum_{j=1}^n \tilde{P}_{ij} D_j.$$

Since all walks start at  $v_1$ , there is always a visit to  $v_1$  which does not have a predecessor. This implies that  $D_1$  is 1 greater than the weighted sum of its neighbors values. For  $i = 1$  we have,

$$D_1 = 1 + \sum_{j=1}^n P_{1j} D_j = \tilde{P}_{1n} D_n + \sum_{j=1}^{n-1} \tilde{P}_{1j} D_j = \sum_{j=1}^n \tilde{P}_{1j} D_j.$$

Then  $D = \tilde{P}D$ , and  $D$  is an eigenvector of  $\tilde{P}$  with eigenvalue 1, and  $D_n = 1$  by assumption. As shown above this vector is unique, so  $D = E$ .  $\square$

This process can be made slightly simpler by using the assumption that  $D_n = 1$  to reduce the dimensionality of the matrices by 1. Define  $\bar{P}$  as the  $(n-1) \times (n-1)$  leading principal submatrix of  $P$ .  $\bar{P}$  is also Similarly, define  $\bar{D}$  as the vector formed by the first  $n-1$  coordinates of  $D$ .

**Theorem 2.**

$$(I - \bar{P})\bar{D} = \chi_1$$

$(I - \bar{P})$  is invertible, so

$$\bar{D} = (I - \bar{P})^{-1}\chi_1$$

*Proof.* By proposition 1,  $(I - \tilde{P})D = 0$ . Writing this in block matrix form for the first  $n - 1$  rows shows that

$$0 = [I - \bar{P} \quad -\chi_1] \cdot \begin{bmatrix} \bar{D} \\ 1 \end{bmatrix} = (I - \bar{P})\bar{D} - \chi_1$$

It suffices to show that  $I - \bar{P}$  is invertible.  $\tilde{P}$  has a one-dimensional eigenspace for the eigenvalue 1 by the Perron-Frobenius theorem, which implies that  $I - \tilde{P}$  has a nullity of 1 and a rank of  $n - 1$ . The Perron-Frobenius eigenvector has all positive coordinates, so any column can be written as a linear combination of all of the other columns. Since each column of  $\tilde{P}$  sums to 1, each column of  $I - \tilde{P}$  sums to 0, so the last row can be written as the negative sum of all the other rows. The removal of the last column, and the last row do not affect the rank of the matrix, so  $I - \bar{P}$  is invertible.  $\square$

The next section examines when a vector  $D$  permits a set of weights  $W$  so that  $D$  is the vector of expected occupation times for  $W$ , and how  $W$  can be computed in that case.

### 2.3 FINDING THE WEIGHTS

By the above work given a set of weights  $W$ ,  $0 = (I - P(W))D - \chi_1$ . Given a set of expected occupation times  $D$ , define  $F(W) = (I - P(W))D - \chi_1$ .  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , so  $F_i$  refers to the  $i$ th component of  $F$ . Any set of weights  $W$  for which  $D$  is the set of expected occupation times will have  $F(W) = 0$ , so the question becomes one of when  $F$  has a zero in  $\mathbb{R}_{>0}^n$ , and how to find it in the case that such a zero exists.

An obvious approach from here would be to use Newton's method to approximate the root. This works sometimes, and converges quadratically, as expected, when such a zero exists. Consider the following example on  $\overline{C_7}$ , the complement of the 7-cycle, where the starting and ending vertices are not adjacent. If all vertices are

given a weight of one, then the expected occupation times are (2.1099, 0.83517, 1.0989, 1.0549, 1.0110, 1.2747, 1). Assume these occupation times for  $D$ , and apply Newton's method to solve for the weights. Then, fixing  $W_n = 1$ , the result of Newton's method is shown in Tables 2.1 and 2.2.

Table 2.1 Newton's Method on  $\overline{C}_7$

1.000000	2.182303	1.000000	1.000000	1.000000	1.000000
0.082406	-0.056562	0.059299	0.150052	0.298105	0.253446
0.037769	0.006201	0.004263	-0.002019	0.173600	0.195977
0.037604	0.030471	0.005884	0.003796	-0.004851	0.026110
0.107309	0.098305	-0.010239	-0.010828	-0.027468	-0.023338
0.057032	0.033218	0.098918	0.100978	0.062051	0.054585
0.290475	0.284835	0.277364	0.277247	0.287089	0.290019
0.618280	0.617896	0.617637	0.617710	0.618096	0.618289
0.890610	0.890601	0.890600	0.890606	0.890614	0.890616
0.991025	0.991025	0.991025	0.991025	0.991025	0.991025
0.999939	0.999939	0.999939	0.999939	0.999939	0.999939

In Table 2.1 the vector is converging to all ones, and seems to be doing so quadratically in each variable. The problem with using Newton's method is that the basins of attraction can be quite small, irregular, and difficult to predict. While it could be useful in some circumstances, this is not a good general method to solve for weights. The reality of this is made more evident by Table 2, where the previous starting point has been changed by a small amount.

Table 2.2 Another Newton's Method on  $\overline{C}_7$

1.000000	2.182304	1.000000	1.000000	1.000000	1.000000
0.082405	-0.056563	0.059298	0.150050	0.298104	0.253445
0.037753	0.006184	0.004247	-0.002035	0.173591	0.195968
0.037176	0.030043	0.005515	0.003426	-0.004999	0.025971
0.099878	0.090665	-0.027041	-0.027633	-0.045961	-0.041732
-0.839434	-0.945517	-0.481074	-0.533826	-1.055920	-1.150283
-2.162853	-2.418231	-2.198071	-1.466400	-1.251899	-1.420593
-4.952298	-4.696043	-4.970810	-5.525099	-5.601439	-5.427250
-27.894333	-27.982593	-27.879736	-27.688253	-27.630389	-27.710694
-621.339288	-621.311739	-621.343273	-621.396474	-621.415349	-621.392844

Convergence is lost completely, and the coordinates are sent off towards infinity. Depending on the starting point, Newton's method can even converge to a set of negative weights. In order to get around the problems with Newton's method, an alternate approach is used, which is to approximate the function with something other than linear functions. For this approach, an algorithm is given to find a good starting position as long as  $D$  satisfies a certain property. It is shown that all  $D$ 's for which a set of weights exists have this property.

The final result on this matter which is proven is a list of equivalent properties for  $D$ . In order to state these properties, a few more definitions are needed. The trace of a walk  $\omega$  at  $v_i$  is defined as  $\text{tr}_i(\omega)$ , the number of times  $\omega$  visits  $v_i$ . Then  $\text{tr}(\omega)$  is the vector giving the number of visits to each vertex. Define the set of proper walks  $\Lambda$  as the set of walks which start at  $v_1$ , end at  $v_n$ , and visit  $v_n$  only once. For  $\omega \in \Lambda$ , let  $p(\omega)$  be the probability of the walk  $\omega$ . A set  $\Omega \subseteq \Lambda$  covers an edge  $e \in G$  if there exists a walk  $\omega \in \Omega$  such that  $\omega$  uses the edge  $e$ .  $\Omega$  covers all edges of  $G$  if such an  $\omega$  exists for each edge in  $G$ . For a set of points  $S \in \mathbb{R}^n$ , the affine hull of  $S$  is defined as the set of points which can be written as  $\sum_{s \in S} \alpha_s s$  with  $\alpha_s \in \mathbb{R}$  for all  $s$ , and  $\sum_{s \in S} \alpha_s = 1$ . A convex combination of elements of  $S$  is a sum  $\sum_{s \in S} \alpha_s s$  with  $\alpha_s \in \mathbb{R}$  for all  $s$ , and  $\sum_{s \in S} \alpha_s = 1$  where  $\alpha_s \geq 0$  for all  $s \in S$ . A strict convex combination is a convex combination for which  $\alpha_s > 0$  for all  $s \in S$ . The convex hull of  $S$  is the subset of the affine hull containing only points which are convex combinations of elements of  $S$ . A point  $x$  in the convex hull is in the relative interior of the convex hull with respect to the affine hull if there exists  $\epsilon > 0$  such that every point in the affine hull which is within  $\epsilon$  of  $x$  is also in the convex hull. By definition,

$$D = \sum_{\omega \in \Lambda} p(\omega) \text{tr}(\omega).$$

$D$  is a convex combination of traces of walks, so it is contained in the convex hull of the traces of proper walks, call this convex hull  $\mathcal{C}$ . The minimal hyperplane in  $\mathbb{R}^n$  which contains  $\mathcal{C}$  is the affine hull of the traces of  $\Lambda$ , call it  $\mathcal{H}$ . Define a hypergraph

$H$  on the same vertices as  $G$  such that  $e \subseteq V(G)$  is an edge in  $H$  iff there exists  $v_k$  such that  $e = N^+(v_k)$ . Let  $c = c(G)$  denote the number of components of  $H$ , and let  $C_i$  be the vertices in the  $i$ th component of  $H$  for  $1 \leq i \leq c$ .  $R \subseteq V$  is a set of representatives of the components, that is  $|R \cap C_i| = 1$ .  $R(v_i)$  is used to mean the representative of the component containing  $v_i$ .  $\xi_S = 1$  if  $v_1 \in S$ , and 0 otherwise for  $S \subseteq V$ , with  $\xi_i = \xi_{\{v_i\}}$ .

$$F_i(W) = D_i - \xi_i - \sum_{j \in N^-(v_i)} D_j \frac{W_i}{\sum_{k \in N^+(v_j)} W_k}.$$

**Theorem 3.** *The following are equivalent*

1. *There exists a set of weights  $W$  such that  $D$  is the vector of expected occupation times on  $G$ .*
2.  *$D$  is in the relative interior of  $\mathcal{C}$  with respect to  $\mathcal{H}$ .*
3. *There exists  $\Omega \subseteq \Lambda$  which covers all edges of  $G$  such that  $D$  is a strict convex combination of the traces of walks in  $\Omega$ .*
4. *For every  $1 \leq i \leq c$  and for every  $S \subseteq C_i$*

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S$$

*with equality iff  $S = C_i$  or  $S = \emptyset$ .*

5. *For every  $S \subseteq V$ ,*

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S$$

*with equality iff  $S = \cup_{i \in \sigma} C_i$  for some  $\sigma \subseteq [c]$*

6. *For every set of representatives,  $R$ , there exists a set of weights  $W$  such that  $F_i(W) \leq 0$  for all  $i \in V - R$ .*

The proof will proceed by showing that each property implies the next one on the list. First, the following property of  $\mathcal{H}$  is proven.

**Theorem 4.**  $\mathcal{H}$  is a hyperplane of dimension  $n - c$ .

*Proof.* The dimension of  $\mathcal{H}$  is equal to the dimension of the span of all possible differences between two points in  $\mathcal{H}$ . The difference between any two points in  $\mathcal{H}$  can be written as  $\sum_{\omega \in \Lambda} \alpha_\omega \text{tr}(\omega)$  where  $\sum_{\omega \in \Lambda} \alpha_\omega = 0$ . Call this space  $\mathcal{H}'$

Let  $B_i = N^-(C_i)$ , and note that  $N^+(B_i) = C_i$ . Note also that  $v_n \notin B_i$  for any  $i$ , since  $N^+(v_n) = \emptyset$ . For any proper walk  $\omega$ , anytime it visits a vertex in  $B_i$  the next vertex must be in  $C_i$ . Anytime a walk visits a vertex in  $C_i$ , the previous vertex must have been in  $B_i$ , unless it is visiting the first vertex. So the number of visits to vertices in  $B_i$  is the same as the number of visits to vertices in  $C_i$ , with one extra step in  $C_i$  if  $v_1 \in C_i$ . That is,

$$\sum_{v_j \in C_i} \text{tr}_j(\omega) = \sum_{v_j \in B_i} \text{tr}_j(\omega) + \xi_{C_i}. \quad (2.1)$$

Let  $\mu_j(C_i)$  be the vector such that  $\mu_j(C_i) = 1$  if  $v_j \in C_i$ ,  $\mu_j(C_i) = -1$  if  $v_j \in B_i$ , and  $\mu_j(C_i) = 0$  otherwise. Then for any proper walk  $\omega \in \Lambda$ ,

$$\mu(C_i) \cdot \text{tr}(\omega) = \xi_{C_i}.$$

For any  $x \in \mathcal{H}$ ,  $\mu(C_i) \cdot x = \xi_{C_i}$ , so for any  $x \in \mathcal{H}'$ ,  $\mu(C_i) \cdot x = 0$ . If these  $\mu$ 's are linearly independent then the dimension of  $\mathcal{H}$  is at most  $n - c$ .

To see that the  $\mu$ 's are linearly independent, note that every vertex in  $G$  shows up in exactly one  $B_i$  and exactly one  $C_i$ , with the exception of  $v_n$ , which is in a  $C_i$  but not a  $B_i$ . In a linear combination of  $\mu$ 's,  $\chi_n$  shows up at most once. If the linear combination equals 0, the coefficient for the component containing  $v_n$  must be 0. But then any vertex with  $v_n$  as a forward neighbor only shows up at most once, so any component with an out-neighbor of  $v_n$  must have a coefficient of 0 as well.

By assumption, every vertex in  $G$  has a path to  $v_n$ , so all of the coefficients must be 0, and therefore the  $\mu$ 's are linearly independent.

To show that  $\mathcal{H}'$  has dimension at least  $n - c$  it suffices to provide  $n - c$  linearly independent vectors in  $\mathcal{H}'$ . Let  $T$  be a rooted spanning tree of  $G$ , rooted at  $v_n$  so that all edges are directed towards  $v_n$ .  $T$  exists because every vertex  $v_i$  has a path to  $v_n$  in  $G$ . Let  $\ell(v_i)$  be the unique path from  $v_i$  to  $v_n$  in  $T$ . Define  $L(v_i) = \text{tr}(\ell(v_i))$ , so the  $L$ 's are clearly linearly independent since the  $\chi_i$ 's can be reconstructed from them. Suppose  $v_i$  and  $v_j$  are neighbors in  $H$ , this implies that there is a  $v_k$  with  $v_k \rightarrow v_i$ , and  $v_k \rightarrow v_j$ . Then any walk from  $v_1$  to  $v_k$ , which must exist by the definition of  $G$ , can be extended to a proper walk either by going to  $v_i$  and following  $\ell(v_i)$ , or doing the same for  $v_j$ . The difference between the traces of these walks is  $L(v_i) - L(v_j)$ . This difference is in  $\mathcal{H}'$  as long as  $v_i$  and  $v_j$  are neighbors in  $H$ . Pick a set of representatives  $R$  for the components. Then, since  $C_i$  is by definition connected,  $L(v_i) - L(R(v_i)) \in \mathcal{H}'$ . This gives a non-zero vector whenever  $v_i \neq R(v_i)$ , so there are  $n - c$  such vectors. Furthermore, any linear combination of these differences with non-zero coefficients gives a linear combination of  $L$ 's with non-zero coefficients, so the differences are linearly independent. The dimension of  $\mathcal{H}'$ , and therefore  $\mathcal{H}$  is  $n - c$ . In addition, the vectors  $L(v_i) - L(R(v_i))$  form a basis for  $\mathcal{H}'$ .  $\square$

As shown above,  $D$  is a convex combination of traces of proper walks. Furthermore, because the coefficient on the trace of a particular walk is the probability of the walk, and each walk has a non-zero probability,  $D$  is a strict convex combination of traces of proper walks. The following lemma proves that property 1 in Theorem 3 implies property 2.

**Lemma 5.** *Let  $S$  be a countable or finite set of points in  $\mathbb{R}^n$ , and suppose  $D$  is a strict convex combination of  $S$ . Then  $D$  is in the relative interior of the convex hull of  $S$  with respect to the affine hull of  $S$ .*

*Proof.*  $D = \sum_{s_i \in S} \alpha_i s_i$ , with  $\sum_{s_i \in S} \alpha_i = 1$ , and  $\alpha_i > 0$  for all  $s_i \in S$ . It suffices to show that there exists an  $\epsilon > 0$  such that for every  $w = \sum_{s_i \in S} a_i s_i$  with  $\sum_{s_i \in S} a_i = 0$  and  $|w| < \epsilon$ ,  $D + w$  is in the convex hull of  $S$ . These  $w$  form a vector space, call it  $\mathcal{W}$ . Let  $U = \{u_1, \dots, u_k\}$  be an orthonormal basis for  $\mathcal{W}$ . Also, note that  $\mathcal{W}$  is in the span of vectors of the form  $s_i - D$ , so there is a finite subset  $S' \subseteq S$  such that  $\mathcal{W}$  is in the span of the linearly independent set of vectors of the form  $s'_i - D$ , with  $s'_i \in S'$ . Then we can write each of the following:

$$\begin{aligned} w &= \sum_{s'_i \in S'} \beta_i (s'_i - D), \\ w &= \sum_{u_i \in U} \gamma_i u_i, \\ u_i &= \sum_{s'_j \in S'} \Gamma_{i,j} (s'_j - D). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{s'_i \in S'} \beta_i (s'_i - D) &= \sum_{u_i \in U} \gamma_i u_i = \sum_{u_i \in U} \gamma_i \sum_{s'_j \in S'} \Gamma_{i,j} (s'_j - D) \\ &= \sum_{s'_j \in S'} (s'_j - D) \sum_{u_i \in U} \gamma_i \Gamma_{i,j}. \end{aligned}$$

Thus  $\beta_j = \sum_{u_i \in U} \gamma_i \Gamma_{i,j}$ . Let  $\Gamma_{\max} = \max_{s'_j \in S'} \sqrt{\sum_{u_i \in U} \Gamma_{i,j}^2}$ . The Cauchy-Schwarz inequality implies that

$$|\beta_j| \leq \sqrt{\sum_{u_i \in U} \gamma_i^2 \cdot \sum_{u_i \in U} \Gamma_{i,j}^2} \leq \epsilon \Gamma_{\max}.$$

Let  $\alpha_{\min} = \min_{s'_i \in S'} \alpha_i$  be the smallest  $\alpha$  amongst the coefficients for elements of  $S'$ .

$$D + w = \sum_{s_i \in S} \alpha_i s_i + \sum_{s'_i \in S'} \beta_i (s'_i - D) = \sum_{s_i \in S} \alpha_i (1 - \sum_{s'_j \in S'} \beta_j) s_i + \sum_{s'_i \in S'} \beta_i s'_i$$

For  $b_i$  such that  $D + w = \sum_{s_i \in S} b_i s_i$ ,

$$b_i \geq \alpha_{\min} (1 - n \epsilon \Gamma_{\max}) - \epsilon \Gamma_{\max} = \alpha_{\min} - \epsilon \Gamma_{\max} (n \alpha_{\min} + 1)$$

For

$$\epsilon = \frac{\alpha_{\min}}{\Gamma_{\max} (n \alpha_{\min} + 1)}$$

$D + w$  is in the convex hull of  $S$ , as desired, when  $|w| < \epsilon$ .  $\square$

*Proof of 2  $\Rightarrow$  3.* Suppose  $D$  is in the relative interior of  $\mathcal{C}$  with respect to  $\mathcal{H}$ . Let  $U$  be a finite set of walks  $u_i$  which covers the edges of  $G$ . Since every edge in  $G$  has a walk from  $v_1$  to  $v_n$  containing it, such a  $U$  exists. Let  $X = \sum_{i=1}^{|U|} \frac{1}{|U|} \text{tr}(u_i)$ , which implies  $D - X \in \mathcal{H}'$ . Since  $D$  is in the relative interior, there exists  $\epsilon > 0$  such that  $D + \epsilon(D - X)$  is also in the relative interior. There exist  $\alpha_i \geq 0$  such that

$$D + \epsilon(D - X) = \sum_{\omega_i \in \Lambda} \alpha_i \text{tr}(\omega_i)$$

$$D = \sum_{\omega_i \in \Lambda} \frac{\alpha_i}{1 + \epsilon} \text{tr}(\omega_i) + \sum_{i=1}^{|U|} \frac{\epsilon}{(1 + \epsilon)|U|} \text{tr}(u_i)$$

$D$  can be written as a strict convex combination of walks which cover every edge in  $G$ , implying  $D$  has property 3.  $\square$

Equation 2.1 implies property 4 holds with equality when  $S = C_i$ . Property 4 also clearly holds when  $S = \emptyset$ . The following lemmas complete the proof of 3  $\Rightarrow$  4.

**Lemma 6.** *For any proper walk  $\omega \in \Lambda$ , for every  $1 \leq i \leq c$  and for every  $S \subseteq C_i$*

$$\sum_{v_j \in S} \text{tr}_j(\omega) \leq \sum_{v_j \in N^-(S)} \text{tr}_j(\omega) + \xi_S$$

*Proof.* Every time  $\omega$  visits a vertex in  $S$ , the previous vertex must be in  $N^-(S)$ , with the exception of the first visit to  $v_1$ . The total number of visits to  $N^-(S)$  is at least the number of visits to  $S$ , minus 1 if  $v_{\text{start}} \in S$ .  $\square$

**Corollary 7.** *For  $D \in \mathcal{C}$ , for every  $1 \leq i \leq c$  and for every  $S \subseteq C_i$*

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S$$

**Lemma 8.** *Suppose  $D$  is a strict convex combination of traces of walks in  $\Omega \subseteq \Lambda$ , and  $\Omega$  covers the edges of  $G$ . Then for every  $1 \leq i \leq c$  and for every non-empty  $S \subsetneq C_i$*

$$\sum_{v_j \in S} D_j < \sum_{v_j \in N^-(S)} D_j + \xi_S$$

*Proof.*  $D = \sum_{\omega_j \in \Omega} \alpha_j \omega_j$ , with  $\alpha_j > 0$ , and  $\sum_{\omega_j \in \Omega} \alpha_j = 1$ . Since  $S \neq C_i$  is non-empty and  $C_i$  is connected in  $H$ , there exists  $x \in S$  and  $y \in C_i - S$  such that  $x$  and  $y$  are neighbors in  $H$ . In other words, there exists  $z$  with  $z \rightarrow x$  and  $z \rightarrow y$ . There exists a walk  $\omega_k \in \Omega$  from  $v_1$  to  $v_n$  which goes through the edge  $zy$ . Since  $\omega_k$  uses the edge  $zy$ , it goes from  $N^-(S)$  to  $C_i - S$  at least once, thus

$$\sum_{v_j \in S} \text{tr}(\omega_k)_j < \sum_{v_j \in N^-(S)} \text{tr}(\omega_k)_j + \xi_S. \quad (2.2)$$

If  $\alpha_k = 1$ , then the proof is complete. Otherwise let

$$E = D + \frac{\frac{1}{2}\alpha_k}{1 - \alpha_k} (D - \text{tr}(\omega_k)) = \sum_{j \neq k} \alpha_j \left( 1 + \frac{\frac{1}{2}\alpha_k}{1 - \alpha_k} \right) \text{tr}(\omega_j) + \frac{1}{2}\alpha_k \text{tr}(\omega_k)$$

$E$  is also a convex combination of walks, so by corollary 7,

$$\sum_{v_j \in S} E_j \leq \sum_{v_j \in N^-(S)} E_j + \xi_S,$$

$$\sum_{v_j \in S} \left( D + \frac{\frac{1}{2}\alpha_k}{1 - \alpha_k} (D - \text{tr}(\omega_k)) \right)_j \leq \sum_{v_j \in N^-(S)} \left( D + \frac{\frac{1}{2}\alpha_k}{1 - \alpha_k} (D - \text{tr}(\omega_k)) \right)_j + \xi_S.$$

If

$$\sum_{v_j \in S} (D - \text{tr}(\omega_k))_j > \sum_{v_j \in N^-(S)} (D - \text{tr}(\omega_k))_j,$$

then we have strict inequality with  $D$ . So now suppose

$$\sum_{v_j \in S} (D - \text{tr}(\omega_k))_j \leq \sum_{v_j \in N^-(S)} (D - \text{tr}(\omega_k))_j.$$

Then, using Equation 2.2,

$$\begin{aligned} \sum_{v_j \in S} D_j &= \sum_{v_j \in S} (D - \text{tr}(\omega_k))_j + \sum_{v_j \in S} \text{tr}(\omega_k)_j \\ &< \sum_{v_j \in N^-(S)} (D - \text{tr}(\omega_k))_j + \sum_{v_j \in N^-(S)} \text{tr}(\omega_k)_j + \xi_S = \sum_{v_j \in N^-(S)} D_j + \xi_S. \end{aligned}$$

□

*Proof of 4  $\Rightarrow$  5.* Pick a set  $S \subseteq V$ , and decompose it into subsets  $S_i \subseteq C_i$ . Applying the inequality in Property 4 to each subset implies,

$$\sum_{v_j \in S_i} D_j \leq \sum_{v_j \in N^-(S_i)} D_j + \xi_{S_i},$$

and then putting the pieces back together proves that

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S.$$

Equality holds iff  $S$  is a union of components. □

Some further properties implied by Property 5 are worth noting.

**Theorem 9.** *Property 5 also implies that  $D_n = 1$ ,  $D_1 \geq 1$ , and for  $i \neq 1, n$ ,  $D_i > 0$ .*

$D_1 = 1$  iff  $N^-(v_1) = \emptyset$ .

*Proof.* Applying Property 5 when  $S = V$  shows that

$$\sum_{v_j \in V} D_j = \sum_{v_j \in N^-(V)} D_j + \xi_S.$$

Every vertex shows up on the left hand side, but  $v_n$  doesn't show up on the right hand side, whereas everything else does.  $D_n = \xi_V = 1$ .

Let  $S$  be the set of vertices  $v_i$  such that  $D_i \leq \xi_i$ . By property 5,

$$\begin{aligned} \sum_{v_i \in V-S} D_i &\leq \sum_{v_i \in N^-(V-S)} D_i + \xi_{V-S} \\ &= \sum_{v_i \in N^-(V-S) \cap (V-S)} D_i + \sum_{v_i \in N^-(V-S) \cap S} D_i + \xi_{V-S} \\ &= \sum_{v_i \in N^-(V-S) \cap (V-S)} D_i + \sum_{v_i \in N^-(V-S) \cap S} D_i + D_n - \xi_S \\ &\leq \sum_{v_i \in V-S} D_i + \sum_{v_i \in N^-(V-S) \cap S} D_i - \xi_S \\ &\leq \sum_{v_i \in V-S} D_i. \end{aligned}$$

All of the inequalities must actually be equalities. By Property 5,  $V - S$  is the union of some collection of components.

$$\sum_{v_i \in N^-(V-S) \cap S} D_i = \xi_S. \quad (2.3)$$

$S$  must also be the union of components, or it is empty. By Property 5,

$$\begin{aligned} \sum_{v_i \in S} D_i &= \sum_{v_i \in N^-(S)} D_i + \xi_S \\ &= \sum_{v_i \in N^-(S) \cap S} D_i + \sum_{v_i \in N^-(S) \cap (V-S)} D_i + \xi_S \\ &= \sum_{v_i \in N^-(S) \cap S} (D_i - \xi_i) + \sum_{v_i \in N^-(S) \cap (V-S)} D_i + \xi_S + \xi_{N^-(S) \cap S} \\ &\geq \sum_{v_i \in S} (D_i - \xi_i) + \sum_{v_i \in N^-(S) \cap (V-S)} D_i + \xi_S + \xi_{N^-(S) \cap S} \\ &= \sum_{v_i \in S} D_i + \sum_{v_i \in N^-(S) \cap (V-S)} D_i + \xi_{N^-(S) \cap S}. \end{aligned}$$

$N^-(S) \cap (V - S) = \emptyset$ , and  $v_1 \notin N^-(S) \cap S$ . If  $v_i \notin S$ , then  $N^+(v_i) \cap S = \emptyset$ , and  $N^+(v_1) \cap S = \emptyset$ . Pick a vertex  $v_j \neq v_1$ , there is a path from  $v_1$  to  $v_j$  but no vertices in the path except  $v_1$  can be in  $S$ , so  $v_j \notin S$ . Either  $S = \emptyset$ , or  $S = \{v_1\}$ . In the case that  $S = \{v_1\}$ ,  $v_1 \in N^-(V - S)$ , so by equation 2.3,  $D_1 = \xi_S = 1$ . This is only possible if  $v_1$  has no in-edges in  $G$ .  $\square$

It is useful to note here that properties 4 and 5 can be replaced by equivalent conditions where the in-neighborhoods have been replaced by the out-neighborhoods in the definitions. That is, define a hypergraph  $H'$  on the vertices of  $G - v_n$  with  $e \subseteq V$  an edge in  $H'$  iff  $e = N^+(v_k)$  for some  $v_k \in V$ . Let  $B_i$  be the components of  $H'$ . Note that, as in the proof of the dimension of  $\mathcal{H}$ ,  $B_i = N^-(C_i)$  and  $C_i = N^+(B_i)$ . The following properties can be included in Theorem 3.

4b. For  $1 \leq i \leq c$  and for every  $T \subseteq B_i$

$$\sum_{v_j \in T} D_j + \xi_{N^+(T)} \leq \sum_{v_j \in N^+(T)} D_j$$

with equality iff  $T = B_i$  or  $T = \emptyset$ .

5b. For every  $T \subseteq V - v_n$ ,

$$\sum_{v_j \in T} D_j + \xi_{N^+(T)} \leq \sum_{v_j \in N^+(T)} D_j$$

with equality iff  $T = \cup_{i \in \sigma} B_i$  for some  $\sigma \subseteq [c]$

We will prove  $4 \Rightarrow 4b$ , but the reverse proof, and the proof for 5 is much the same.

*proof of  $4 \Rightarrow 4b$ .* Let  $S = C_i - N^+(T)$ . Note that  $N^-(S) \subseteq B_i - T$ .

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S \tag{2.4}$$

Theorem 9 shows that

$$\begin{aligned} \sum_{v_j \in C_i} D_j - \sum_{v_j \in N^+(T)} D_j &\leq \sum_{v_j \in N^-(S)} D_j + \xi_S \\ &\leq \sum_{v_j \in B_i} D_j + \xi_{C_i} - \sum_{v_j \in T} D_j - \xi_{N^+(T)}. \end{aligned}$$

Applying Property 4 to  $C_i$  implies

$$\sum_{v_j \in T} D_j + \xi_{N^+(T)} \leq \sum_{v_j \in N^+(T)} D_j,$$

with equality iff Equation 2.4 has equality and  $N^-(S) = B_i - T$ . These are both true when  $T = \emptyset$  and when  $T = B_i$ , so it suffices to show that they don't hold otherwise. If  $N^+(T) = C_i$  or  $N^+(T) = \emptyset$ , then Equation 2.4 holds with equality. If  $N^+(T) = \emptyset$ , then  $T = \emptyset$ , since  $v_n$  is the only sink in  $G$ . If  $N^+(T) = C_i$ , then  $S = \emptyset$  and  $B_i = T$ .  $\square$

The proof of  $5 \Rightarrow 6$  utilizes Algorithm A to construct the set of weights  $W$  satisfying the conditions of Property 6,  $F_j(W) \leq 0$  for all  $j \notin R$ . The algorithm initializes all of the weights at 1. From here, it checks which  $F_j$ 's are positive, and multiplies the corresponding weights by  $1 + \tau$ , for some given  $\tau > 0$ , if  $j \notin R$ . This process repeats as long as at least one weight was increased, continuing until all of the desired  $F_j$ 's are non-positive. Based on the definition of Algorithm A, it is not

Algorithm A Pseudocode

```

1   for  $j \leftarrow 1$  to  $n$ 
2        $W_j(0) \leftarrow 1$ 
3    $k \leftarrow 0$ 
4   repeat
5        $t \leftarrow 0$ 
6       for  $j \leftarrow 1$  to  $n$ 
7           if  $j \notin R$  and  $F_j(W(k)) > 0$ 
8               then  $W_j(k+1) \leftarrow (1 + \tau)W_j(k)$ 
9                    $t \leftarrow 1$ 
10              else  $W_j(k+1) \leftarrow W_j(k)$ 
11           $k \leftarrow k + 1$ 
12  until  $t = 0$ 
13  return  $W(k)$ 

```

clear that the algorithm will terminate. Whether or not it does depends on the choice of  $F$ ,  $\tau$ , and  $R$ . In order to prove that  $5 \Rightarrow 6$  for the proof of Theorem 3, it suffices to show that when  $F$  is defined by a vector  $D$  which satisfies Property 5, then Algorithm A will end in finitely many steps for the correct choice of  $\tau$ .

The following theorem gives a pair of properties which, if satisfied, provide conditions under which Algorithm A will always end. A few definitions are needed to state these properties. For  $S \subseteq [n] - R$ ,  $\eta > 0$ , define  $\mathcal{Y}_S(\eta) \subseteq \mathbb{R}_{>0}^n$  as the set of  $W \in \mathbb{R}_{>0}^n$  such that

$$\frac{\min_{i \in S} W_i}{\sum_{i \in [n] - S} W_i} > \eta$$

**Theorem 10.** *Suppose there exists  $0 < \delta_1 < \delta_2$  such that the following properties hold for every  $S \subseteq [n] - R$ ,*

1.

$$\liminf_{\eta \rightarrow \infty} \left( \sup_{W \in \mathcal{Y}_S(\eta)} \sum_{i \in S} F_i(W) \right) \leq -\delta_2 |S|.$$

2. *If*

$$F_i(W(k)) > 0 \quad \text{when } i \in S$$

$$F_i(W(k)) \leq 0 \quad \text{when } i \notin S$$

*then*

$$F_i(W(k+1)) > -\delta_1 \quad \text{when } i \in S$$

$$F_i(W(k+1)) \geq F_i(W(k)) \quad \text{when } i \notin S$$

*Then Algorithm A ends in finitely many steps.*

*Proof.* This proof is by contradiction; suppose the algorithm never halts. Then some  $W$ 's have their weights increased infinitely often, let  $S$  be the indices of these  $W$ 's.

$$\lim_{k \rightarrow \infty} \frac{\min_{i \in S} W_i(k)}{\sum_{i \in [n] - S} W_i(k)} = \infty$$

For any  $i \in S$ ,

$$\liminf_{k \rightarrow \infty} F_i(W(k)) \geq -\delta_1.$$

$$-\delta_1 |S| \leq \sum_{i \in S} \liminf_{k \rightarrow \infty} F_i(W(k)) \leq \liminf_{k \rightarrow \infty} \sum_{i \in S} F_i(W(k)) \leq -\delta_2 |S|.$$

This is a contradiction since  $\delta_1 < \delta_2$ , so the algorithm must end after finitely many steps. □

Let  $R$  be a set of representatives of the components.  $R(v_j)$  will be used to denote the representative of the component  $C_i$  such that  $v_j \in C_i$ . The algorithm can execute separately on the components  $C_i$ , so the following parameters are defined for each component.  $\delta_1$  and  $\tau$  will involve a parameter  $\epsilon_i \in (0, 1)$ . In order to satisfy Theorem 10,  $\epsilon_i$  cannot be 0 or 1, but any choice between 0 and 1 will work. The effect of  $\epsilon_i$  is

on the number of steps required for the algorithm, which will be discussed in a later section.  $\mu_i$  is defined as

$$\mu_i = \min_{S \subseteq C_i - R_i} \left( \frac{1}{|S|} \left( \sum_{t \in N^-(S)} D_t - \sum_{s \in S} (D_s - \xi_s) \right) \right),$$

and  $\lambda_i = \frac{\sum_{v_j \in B_i} D_j}{\mu_i}$ . Finally, define  $\tau_i = \frac{1 - \epsilon_i}{\lambda_i}$ .

**Lemma 11.** *Given Property 5 from Theorem 3, for every  $S \subseteq C_i - R_i$ ,  $W \in \mathcal{Y}_S(\eta)$ ,*

$$\sum_{j \in S} F_j(W) < -\mu_i |S| + \frac{\sum_{v_j \in B_i} D_j}{\eta + 1}.$$

*Proof.* Let  $S' = C_i - S$

$$\begin{aligned} \sum_{j \in S} F_j(W) &= \sum_{v_j \in S} (D_j - \xi_j) - \sum_{v_j \in S} \sum_{v_\ell \in N^-(v_j)} D_\ell \frac{W_j}{\sum_{v_h \in N^+(v_\ell)} W_h} \\ &= \sum_{v_j \in S} (D_j - \xi_j) - \sum_{v_\ell \in N^-(S)} D_\ell \frac{\sum_{v_j \in N^+(v_\ell) \cap S} W_j}{\sum_{v_h \in N^+(v_\ell)} W_h} \\ &= \sum_{v_j \in S} (D_j - \xi_j) - \sum_{v_\ell \in N^-(S)} D_\ell \left( 1 - \frac{\sum_{v_j \in N^+(v_\ell) \cap S'} W_j}{\sum_{v_h \in N^+(v_\ell)} W_h} \right) \\ &\leq -\mu_i |S| + \sum_{v_\ell \in N^-(S)} D_\ell \frac{\sum_{v_j \in N^+(v_\ell) \cap S'} W_j}{\sum_{v_h \in N^+(v_\ell) \cap S} W_h + \sum_{v_h \in N^+(v_\ell) \cap S'} W_h} \\ &\leq -\mu_i |S| + \sum_{v_\ell \in N^-(S)} D_\ell \frac{\sum_{v_j \in S'} W_j}{\min_{v_h \in S} W_h + \sum_{v_j \in S'} W_j} \\ &< -\mu_i |S| + \frac{\sum_{v_j \in B_i} D_j}{\eta + 1} \end{aligned}$$

□

**Lemma 12.** *Given Property 5 from Theorem 3, for every  $S \subseteq C_i - R_i$ , if*

$$F_j(W(k)) > 0 \quad \text{when } j \in S$$

$$F_j(W(k)) \leq 0 \quad \text{when } j \notin S$$

then

$$F_j(W(k+1)) > -(1 - \epsilon_i)\mu_i \quad \text{when } j \in S$$

$$F_j(W(k+1)) \geq F_j(W(k)) \quad \text{when } j \notin S$$

*Proof.* First, suppose without loss of generality that  $1 \notin S$ , so  $F_1(W(k)) \leq 0$ . This implies  $W_1(k+1) = W_1(k)$ , and  $W_j(k+1) \geq W_j(k)$  for all  $j$ .

$$\begin{aligned} F_1(W(k+1)) &= D_1 - \xi_1 - \sum_{v_j \in N^-(v_1)} D_j \frac{W_1(k+1)}{\sum_{v_h \in N^+(v_j)} W_h(k+1)} \\ &= D_1 - \xi_1 - \sum_{v_j \in N^-(v_1)} D_j \frac{W_1(k)}{\sum_{v_h \in N^+(v_j)} W_h(k+1)} \\ &\geq D_1 - \xi_1 - \sum_{v_j \in N^-(v_1)} D_j \frac{W_1(k)}{\sum_{v_h \in N^+(v_j)} W_h(k)} = F_1(W(k)) \end{aligned}$$

Suppose without loss of generality that  $1 \in S$ , so  $F_1(W(k)) > 0$ . This implies  $W_1(k+1) = (1 + \tau_i)W_1(k)$ , and  $W_j(k+1) \geq W_j(k)$  for all  $j$ .

$$\begin{aligned} F_1(W(k+1)) &= D_1 - \xi_1 - \sum_{v_j \in N^-(v_1)} D_j \frac{W_1(k+1)}{\sum_{v_h \in N^+(v_j)} W_h(k+1)} \\ &= D_1 - \xi_1 - \sum_{v_j \in N^-(v_1)} D_j \frac{(1 + \tau_i)W_1(k)}{\sum_{v_h \in N^+(v_j)} W_h(k+1)} \\ &\geq D_1 - \xi_1 - \sum_{v_j \in N^-(v_1)} D_j \frac{W_1(k)}{\sum_{v_h \in N^+(v_j)} W_h(k)} - \tau_i \sum_{v_j \in N^-(v_1)} D_j \frac{W_1(k)}{\sum_{v_h \in N^+(v_j)} W_h(k)} \\ &\geq F_1(W(k)) - \tau_i \sum_{v_j \in N^-(v_1)} D_j > -\frac{1 - \epsilon_i}{\lambda_i} \sum_{v_j \in B_i} D_j \\ &= -(1 - \epsilon_i)\mu_i \end{aligned}$$

□

**Corollary 13.** *If  $D$  satisfies Property 5, then Algorithm A can be executed in a finite number of steps on  $C_i$  with any choice of representative  $R_i$ .*

*Proof.* By Lemma 11,  $\delta_2 = \mu_i$ . By Lemma 12,  $\delta_1 = (1 - \epsilon)\mu_i$ .  $0 < \delta_1 < \delta_2$ , so Theorem 10 applies.  $\square$

Given  $R$ , Algorithm A produces the requisite set of weights  $W$  for Property 6. A second algorithm, Algorithm B, uses this set of weights as a starting point, and converges to the set of weights in Property 1.

In order to do this an Algorithm B is similar to Newton's method; it will give a sequence of weights, converging to a solution of the equation  $F(W) = 0$ . It is similar to Newton's method in the sense that  $F$  will be approximated by a simpler function,  $G$  and a zero of  $G$  will be found. Then the zero of  $G$  is used as a starting point for a new approximation. There are some problems with just using a straight forward Newton's method, as it is difficult to find a good starting point. This modification will fix this by converging as long as the starting set of weights satisfies Property 6. It still has some of the same problems as Newton's method; it does not converge everywhere to the point we would like, and the boundaries of the basins of attraction are chaotic, but the modified version has the advantage of having a constructable starting point which leads to a converging sequence to a zero of  $F$ .

A slightly modified version of  $F$  is used to reduce dimensionality. First,

$$\begin{aligned}
& \sum_{v_i \in C_j} F_i(W) \\
&= \sum_{v_i \in C_j} (D_i - \xi_i) - \sum_{v_i \in C_j} \sum_{v_k \in N^-(v_i)} D_k \frac{W_i}{\sum_{v_h \in N^+(v_h)} W_h} \\
&= \sum_{v_i \in C_j} (D_i - \xi_i) - \sum_{v_k \in B_j} D_k \frac{\sum_{v_i \in N^+(v_k) \cap C_j} W_i}{\sum_{v_h \in N^+(v_h)} W_h} \\
&= \sum_{v_i \in C_j} (D_i - \xi_i) - \sum_{v_k \in B_j} D_k = 0.
\end{aligned}$$

Given a set of representatives  $R$  of the components  $C_i$ ,  $F$  is uniquely determined by  $F|_{V-R}$ . Secondly, if the weight of every vertex in  $C_i$  is multiplied by the same constant, then the probability matrix  $P$ , and therefore the  $F_j$  functions, don't change. All of the weights are positive, so assume without loss of generality that  $W_i = 1$  if  $v_i \in R$ .  $\bar{F} : \mathbb{R}^{n-c} \rightarrow \mathbb{R}^{n-c}$  is defined for  $v_i \in V - R$ ,

$$\bar{F}_i(W) = D_i - \xi_i - \sum_{v_j \rightarrow v_i} D_j \frac{W_i}{\sum_{v_k \in N^+(v_j) \cap (V-R)} W_k + \beta_j},$$

where  $\beta_j$  is 1 if  $R(v_i) \in N^+(v_j)$ , and 0 otherwise.

The following theorems, which set the stage for Algorithm B, deal with functions similar to  $\bar{F}$  which have several important properties. For  $1 \leq i \leq n$ , say  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$  has property  $\mathbb{P}_i$  if the following conditions hold:

1.

$$F_i(X) = A_i - \sum_{j=1}^{m_i} \frac{X_i}{\sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i}$$

2.  $A_i > 0$ .

3.  $\alpha_{j,k}^i \geq 0$  for all  $1 \leq j \leq m_i$  and  $1 \leq k \leq n + 1$ .

4.  $\alpha_{j,i}^i > 0$  for all  $1 \leq j \leq m_i$ .

5. For every  $1 \leq j \leq m_i$ , there exists at least one  $1 \leq k \leq n + 1$ ,  $k \neq i$  such that  $\alpha_{j,k}^i > 0$ .

Consider  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that each  $F_i$  has property  $\mathbb{P}_i$ . Define the directed graph  $\mathbb{G}_F$  on  $n + 1$  vertices,  $\mathbb{V}_i$ , so that there is an edge from  $\mathbb{V}_i$  to  $\mathbb{V}_k$  iff in  $F_i$  there exists  $j$  such that  $\alpha_{j,k}^i > 0$ . If every vertex  $\mathbb{V}_i$  has a directed path to  $\mathbb{V}_{n+1}$ , say that  $F$  has property  $\mathbb{P}$ . Let  $T$  be a rooted spanning tree in  $\mathbb{G}_F$ , which is rooted at  $\mathbb{V}_{n+1}$ , and all of the edges point towards  $\mathbb{V}_{n+1}$ . Let  $N_T(i)$  be the unique forward neighbor in  $T$  of

$\forall_i$ . Let  $\mathcal{T}_{n+1} = 1$ ;  $\mathcal{T}_i$  is defined recursively. For  $t \in \mathbb{R}$ , define  $\mathcal{Q}^i(t) \in \mathbb{R}^n$  as the vector with  $\mathcal{Q}_i^i(t) = t$ ,  $\mathcal{Q}_j^i(t) = 0$  for  $j \neq i, N_T(i)$ , and if  $N_T(i) \neq n+1$ ,  $\mathcal{Q}_{N_T(i)}^i(t) = \mathcal{T}_{N_T(i)}$ .

$$F_i(\mathcal{Q}^i(t)) = A_i - \sum_{j=1}^{m_i} \frac{t}{\alpha_{j,N_T(i)}^i \mathcal{T}_{N_T(i)} + \alpha_{j,i}^i t}$$

For  $t \geq 0$ , this is a strictly decreasing function with  $F_i(\mathcal{Q}^i(0)) = A_i > 0$ , so there exists a unique  $\mathcal{T}_i > 0$  with  $F_i(\mathcal{Q}^i(\mathcal{T}_i)) = 0$ .

**Lemma 14.** *Let  $F$  be a function with property  $\mathbb{P}$ ,  $Y \in \mathbb{R}_{>0}^n$ , with  $F_i(Y) \leq 0$ , and  $N_T(i) = n+1$  or  $Y_{N_T(i)} \geq \mathcal{T}_{N_T(i)}$ , then  $Y_i \geq \mathcal{T}_i$ .*

*Proof.* This proof is by contrapositive. Suppose  $N_T(i) = n+1$  or  $Y_{N_T(i)} \geq \mathcal{T}_{N_T(i)}$ , and  $Y_i < \mathcal{T}_i$ .

$$\begin{aligned} F_i(Y) &= A_i - \sum_{j=1}^{m_i} \frac{Y_i}{\sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i} \\ &\geq A_i - \sum_{j=1}^{m_i} \frac{Y_i}{\alpha_{j,N_T(i)}^i \mathcal{T}_{N_T(i)} + \alpha_{j,i}^i Y_i} \\ &> A_i - \sum_{j=1}^{m_i} \frac{\mathcal{T}_i}{\alpha_{j,N_T(i)}^i \mathcal{T}_{N_T(i)} + \alpha_{j,i}^i \mathcal{T}_i} = F_i(\mathcal{Q}^i(\mathcal{T}_i)) = 0 \end{aligned}$$

□

**Corollary 15.** *If  $F_i(Y) \leq 0$  for every  $i$ , then  $Y_i \geq \mathcal{T}_i$  for every  $i$ .*

**Proposition 16.** *Fix  $i$  between 1 and  $n$ . Suppose  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$  has property  $\mathbb{P}_i$ . Then for every  $Y \in \mathbb{R}_{>0}^n$  there exists*

$$G_{i,Y}(X) = A_i - \frac{X_i}{\sum_{k=1}^n \theta_k^i X_k + \theta_{n+1}^i}$$

*with property  $\mathbb{P}_i$ , such that  $F_i(Y) = G_{i,Y}(Y)$ , and  $\frac{\partial F_i}{\partial X_j}(Y) = \frac{\partial G_{i,Y}}{\partial X_j}(Y)$  for all  $j$  such that  $1 \leq j \leq n$ .*

*Proof.* Let  $K_j = \frac{\partial F}{\partial X_j}(Y)$ , and let  $B_i = \frac{1}{A_i - F_i(Y)}$ . For  $j \neq i, n+1$ , define

$$\theta_j^i = K_j Y_i (B_i)^2$$

$$\theta_i^i = K_i Y_i (B_i)^2 + B_i = Y_i (B_i)^2 \left( K_i + \frac{1}{Y_i B_i} \right)$$

$$\theta_{n+1}^i = B_i Y_i - \sum_{k=1}^n \theta_k Y_k = -Y_i (B_i)^2 \sum_{k=1}^n K_k Y_k$$

Given these values for the  $\theta$  terms,

$$\sum_{k=1}^n \theta_k^i Y_k + \theta_{n+1}^i = \sum_{k=1}^n \theta_k^i Y_k + B_i Y_i - \sum_{k=1}^n \theta_k^i Y_k = B_i Y_i$$

$$\begin{aligned} G_{i,Y}(Y) &= A_i - \frac{Y_i}{\sum_{k=1}^n \theta_k^i Y_k + \theta_{n+1}^i} = A_i - \frac{Y_i}{B_i Y_i} \\ &= A_i - (A_i - F_i(Y)) = F_i(Y) \end{aligned}$$

For  $j \neq i, n+1$ ,

$$\begin{aligned} \frac{\partial G_{i,Y}}{\partial X_j}(Y) &= \frac{\theta_j^i Y_i}{\left( \sum_{k=1}^n \theta_k^i Y_k + \theta_{n+1}^i \right)^2} = \frac{K_j (Y_i B_i)^2}{(B_i Y_i)^2} = K_j \\ \frac{\partial G_{i,Y}}{\partial X_i}(Y) &= \frac{\theta_i^i Y_i}{\left( \sum_{k=1}^n \theta_k^i Y_k + \theta_{n+1}^i \right)^2} - \frac{1}{\sum_{k=1}^n \theta_k^i Y_k + \theta_{n+1}^i} \\ &= \frac{K_i (Y_i B_i)^2 + Y_i B_i}{(Y_i B_i)^2} - \frac{1}{Y_i B_i} = K_i + \frac{1}{Y_i B_i} - \frac{1}{Y_i B_i} = K_i \end{aligned}$$

Noting that the  $Y_j$  are all positive, to show  $\theta_j \geq 0$  for  $j \neq i, n+1$  it suffices to show that  $K_j \geq 0$ .

$$K_j = \frac{\partial F_i}{\partial X_j}(Y) = \sum_{\ell=1}^{m_i} \frac{\alpha_{\ell,j}^i Y_i}{\left( \sum_{k=1}^n \alpha_{\ell,k}^i Y_k + \alpha_{\ell,n+1}^i \right)^2}$$

Since  $\alpha_{\ell,j}^i \geq 0$ ,  $K_j \geq 0$ , and any  $\alpha_{\ell,j}^i > 0$  would imply  $K_j > 0$ . To prove  $\theta_i > 0$  it suffices to show that  $K_i + (Y_i B_i)^{-1} > 0$ .

$$K_i = \frac{\partial F_i}{\partial X_i}(Y) = \sum_{j=1}^{m_i} \frac{\alpha_{j,i}^i Y_i}{\left( \sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i \right)^2} - \sum_{j=1}^{m_i} \frac{1}{\sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i}$$

$$\frac{1}{Y_i B_i} = \frac{A_i - F_i(Y)}{Y_i} = \sum_{j=1}^{m_i} \frac{1}{\sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i}$$

$$K_i + \frac{1}{Y_i B_i} = \sum_{j=1}^{m_i} \frac{\alpha_{j,i}^i Y_i}{\left( \sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i \right)^2} > 0$$

To prove  $\theta_{n+1}^i \geq 0$ , note that it suffices to show that  $\sum_{k=1}^n K_k Y_k \leq 0$

$$\begin{aligned} \sum_{k=1}^n K_k Y_k &= - \sum_{j=1}^{m_i} \frac{\sum_{k \neq i} \alpha_{j,k}^i Y_i Y_k + \alpha_{j,n+1}^i Y_i}{\left( \sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i \right)^2} + \sum_{\ell \neq i} \sum_{j=1}^{m_i} \frac{\alpha_{j,\ell}^i Y_i Y_\ell}{\left( \sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i \right)^2} \\ &= - \sum_{j=1}^{m_i} \frac{\alpha_{j,n+1}^i Y_i}{\left( \sum_{k=1}^n \alpha_{j,k}^i Y_k + \alpha_{j,n+1}^i \right)^2} \leq 0 \end{aligned}$$

$\theta_{n+1}^i > 0$  if  $\alpha_{j,n+1}^i > 0$  for at least one  $j$ . □

**Corollary 17.** *The  $G_{i,Y}(X)$  guaranteed in Proposition 16 is continuous in both  $X$  and  $Y$  for  $X, Y \in \mathbb{R}_{>0}^n$ , as is  $\frac{\partial G_{i,Y}}{\partial X_j}(X)$ .*

**Lemma 18.** *Suppose  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a function such that  $G_i$  has property  $\mathbb{P}_i$  for every  $i$ , with  $m_i = 1$ . If there exists  $Y > 0$  such that  $G_i(Y) \leq 0$  for all  $i$ , then there exists  $Z$  such that  $0 < Z_i \leq Y_i$  for all  $i$  and  $G_i(Z) = 0$  for all  $i$ .*

*Proof.* The proof is by induction on  $n$ . First, suppose  $n = 1$ . Then

$$G(x) = A - \frac{x}{\theta_1 x + \theta_2},$$

$\theta_1, \theta_2 > 0$ , and  $G(y) \leq 0$ . Since  $G(0) = A > 0$ , and  $G$  is continuous on  $[0, y]$ ,  $z$  exists by the Intermediate Value Theorem.

Now suppose  $n > 1$ .

$$A_n - G_n(Y) = \frac{Y_n}{\sum_{j=1}^n \theta_j^n Y_j + \theta_{n+1}^n}.$$

Using the fact that at least one  $\theta_j^n > 0$ ,  $j \neq n$ ,

$$\begin{aligned} \sum_{j=1}^n \theta_j^n Y_j + \theta_{n+1}^n &> \theta_n^n Y_n \\ Y_n = (A_n - G_n(Y)) \left( \sum_{j=1}^n \theta_j^n Y_j + \theta_{n+1}^n \right) &> (A_n - G_n(Y)) \theta_n^n Y_n \\ 1 - A_n \theta_n^n &> -G_n(Y) \theta_n^n \geq 0 \end{aligned}$$

If  $G_n(X) = 0$ , then

$$X_n = \sum_{j=1}^{n-1} \frac{A_n \theta_j^n}{1 - A_n \theta_n^n} X_j + \frac{A_n \theta_{n+1}^n}{1 - A_n \theta_n^n}$$

Letting  $B_j = \frac{A_n \theta_j^n}{1 - A_n \theta_n^n}$ ,

$$X_n = \sum_{j=1}^{n-1} B_j X_j + B_{n+1}.$$

This induces a map  $Q : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ , where  $Q_i(X) = X_i$  for  $1 \leq i \leq n-1$ , and  $Q_n(X) = \sum_{j=1}^{n-1} B_j X_j + B_{n+1}$ . Then we can define  $\tilde{G} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  such that for any  $i$  with  $1 \leq i < n$ ,  $\tilde{G}_i(X) = G_i(Q(X))$ . For  $X \in \mathbb{R}^{n-1}$ ,

$$\begin{aligned} \tilde{G}_i(X) &= A_i - \frac{Q_i(X)}{\sum_{j=1}^n \theta_j^i Q_j(X) + \theta_{n+1}^i} \\ &= A_i - \frac{X_i}{\sum_{j=1}^{n-1} \theta_j^i X_j + \theta_{n+1}^i + \theta_n^i \left( \sum_{j=1}^{n-1} B_j X_j + B_{n+1} \right)} \\ &= A_i - \frac{X_i}{\sum_{j=1}^{n-1} (\theta_j^i + \theta_n^i B_j) X_j + \theta_{n+1}^i + \theta_n^i B_{n+1}} \end{aligned}$$

Let  $\tilde{Y}$  be the vector formed by the first  $n-1$  coordinates of  $Y$ . Define

$$g(t) = G_n(\tilde{Y}, t) = A_n - \frac{t}{\sum_{j=1}^{n-1} \theta_j^n Y_j + \theta_n^n t + \theta_{n+1}^n}.$$

$g$  is a decreasing function with  $g(Q_n(\tilde{Y})) = 0$ , and  $g(Y_n) = G_n(Y) \leq 0$ .  $Q_n(\tilde{Y}) \leq Y_n$ .

$$\tilde{G}_i(\tilde{Y}) = A_i - \frac{Q_i(\tilde{Y})}{\sum_{j=1}^n \theta_j^i Q_j(\tilde{Y}) + \theta_{n+1}^i}$$

$$\begin{aligned}
&= A_i - \frac{Y_i}{\sum_{j=1}^{n-1} \theta_j^i Y_j + \theta_n^i Q_n(\tilde{Y}) + \theta_{n+1}^i} \\
&\leq A_i - \frac{Y_i}{\sum_{j=1}^{n-1} \theta_j^i Y_j + \theta_n^i Y_n + \theta_{n+1}^i} = G_i(Y) \leq 0
\end{aligned}$$

$\tilde{G}$  and  $\tilde{Y}$  satisfy the conditions for the lemma with  $n - 1$ , so by induction there exists  $\tilde{Z} \in \mathbb{R}^{n-1}$  such that  $0 < \tilde{Z}_i \leq \tilde{Y}_i$  for  $1 \leq i \leq n - 1$ . Let  $Z = Q(\tilde{Z})$ .  $0 < Z_i \leq Y_i$  for  $1 \leq i \leq n - 1$ .  $Z_n = Q_n(\tilde{Z}) \leq Q_n(\tilde{Y}) \leq Y_n$ .  $\square$

**Lemma 19.** Fix  $i$  between 1 and  $n$ . Suppose  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$  has property  $\mathbb{P}_i$ . Let  $G_{i,Y} : \mathbb{R}^n \rightarrow \mathbb{R}$  be the approximation of  $F$  at  $Y$ , as guaranteed by Proposition 16. Then for all  $Z$  with  $0 < Z \leq Y$ ,  $F_i(Z) \leq G_{i,Y}(Z)$ .

*Proof.* Pick  $Z \in \mathbb{R}^n$  with  $0 < Z_i \leq Y_i$  for all  $1 \leq i \leq n$ . Let  $L(t) = (Y - Z)t + Z$ , so that  $L(0) = Z$ , and  $L(1) = Y$ . Let  $M(t) = \frac{L_i(t)}{A_i - F_i(L(t))}$ , and  $N(t) = \frac{L_i(t)}{A_i - G_{i,Y}(L(t))} = \sum_{j=1}^n \theta_j^i L_j(t) + \theta_{n+1}^i$ . It suffices to show that  $M(t) \leq N(t)$  for  $t = 0$ .  $M(1) = N(1)$ , and  $M'(1) = N'(1)$ , so since  $N$  is linear it is the tangent line to  $M$  at 1. It suffices to show that  $M''(t) \leq 0$  for  $t \geq 0$ . There exist  $a_j$  and  $b_j$  such that  $M$  can be written in the form:

$$\begin{aligned}
M(t) &= \frac{L_i(t)}{A_i - F_i(L(t))} = \frac{L_i(t)}{\sum_{j=1}^{m_i} \frac{L_i(t)}{a_j t + b_j}} = \frac{1}{\sum_{j=1}^{m_i} \frac{1}{a_j t + b_j}} \\
M'(t) &= \frac{\sum_{j=1}^{m_i} \frac{a_j}{(a_j t + b_j)^2}}{\left( \sum_{j=1}^{m_i} \frac{1}{a_j t + b_j} \right)^2} \\
M''(t) &= \frac{-2}{\left( \sum_{j=1}^{m_i} \frac{1}{a_j t + b_j} \right)^3} \left( \sum_{j=1}^{m_i} \frac{a_j^2}{(a_j t + b_j)^3} \sum_{j=1}^{m_i} \frac{1}{a_j t + b_j} - \left( \sum_{j=1}^{m_i} \frac{a_j}{(a_j t + b_j)^2} \right)^2 \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{-2}{\left(\sum_{j=1}^{m_i} \frac{1}{a_j t + b_j}\right)^3} \sum_{j=1}^{m_i-1} \sum_{k=j+1}^{m_i} \left( \frac{a_j^2}{(a_j t + b_j)^3} \frac{1}{a_k t + b_k} \right. \\
&\quad \left. - 2 \frac{a_j a_k}{(a_j t + b_j)^2 (a_k t + b_k)^2} + \frac{1}{a_j t + b_j} \frac{a_k^2}{(a_k t + b_k)^3} \right) \\
&= \frac{-2}{\left(\sum_{j=1}^{m_i} \frac{1}{a_j t + b_j}\right)^3} \sum_{j=1}^{m_i-1} \sum_{k=j+1}^{m_i} \left( \frac{1}{(a_j t + b_j)(a_k t + b_k)} \left( \frac{a_j}{a_j t + b_j} - \frac{a_k}{a_k t + b_k} \right)^2 \right)
\end{aligned}$$

If  $t \geq 0$ , then  $M''(t) \leq 0$ . □

Using these lemmas, Algorithm B can now be described. The algorithm initializes with  $Y(0)$ , the output from Algorithm A.  $Y(k)$  is formed from  $Y(k-1)$  by finding the approximation  $G_{Y(k-1)}$ , and solving for a zero of the function.  $G_{Y(k-1)}(Y(k)) = 0$ . This  $G$  exists by Proposition 16. This process continues until whatever level of accuracy is desired has been achieved. The following theorem shows that the sequence formed will converge to a zero of  $F$  if  $F$  has Property  $\mathbb{P}$ .

Algorithm B Pseudocode

```

1    $Y(0) \leftarrow$  output from Algorithm A
2    $k \leftarrow 0$ 
3   repeat
4       approximate  $F(W)$  with  $G_{Y(k)}(W)$ 
5        $Y(k+1) \leftarrow W$  such that  $G_{Y(k)}(W) = 0$ 
6        $k \leftarrow k + 1$ 
7   until convergence criteria met.

```

**Theorem 20.** *Suppose  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  has Property  $\mathbb{P}$ . Let  $Y(0)$  be a point such that  $F_i(Y(0)) < 0$  for all  $i$ . Let  $G_Y : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the function such that  $G_{i,Y}$  is the approximation of  $F_i$  guaranteed by Proposition 16. Then Algorithm B converges to a point  $\bar{Y} \in \mathbb{R}_{>0}^n$  with  $F(\bar{Y}) = 0$ .*

*Proof.* By Lemma 18,  $Y_i(k+1) \leq Y_i(k)$  for all  $1 \leq i \leq n$ . By Lemma 19,

$$F(Y(k+1)) \leq G_{Y(k)}(Y(k+1)) = 0.$$

Each coordinate  $Y_i(k)$  is a non-increasing sequence bounded below by  $\mathcal{T}_i$ , so it converges in  $\mathbb{R}_{>0}^n$ .  $\bar{Y} = \lim_{k \rightarrow \infty} Y(k)$ . By Corollary 17,  $G_{i,Y}(X)$  is continuous in both  $X$  and  $Y$ , so

$$F_i(\bar{Y}) = G_{i,\bar{Y}}(\bar{Y}) = \lim_{k \rightarrow \infty} G_{i,Y(k)}(Y(k+1)) = 0.$$

□

Since  $\bar{F}$  has Property  $\mathbb{P}$ , this proves that  $6 \Rightarrow 1$ , and thus completes the proof of Theorem 3.

## 2.4 NUMBER OF STEPS FOR ALGORITHM A

Recall the following relevant definitions.  $C_i$  is a component of the hypergraph  $H$  where the edges of  $H$  are the out-neighborhoods of  $G$ .  $R$  is a set of representatives of the components,  $R \cap C_i = \{R_i\}$ .  $R(v_j) = R_i$  if  $v_j \in C_i$ .

$$\mu_i = \min_{S \subseteq C_i - R} \left( \frac{1}{|S|} \left( \sum_{t \in N^-(S)} D_t - \sum_{s \in S} (D_s - \xi_s) \right) \right)$$

$$\lambda_i = \frac{\sum_{v_j \in B_i} D_j}{\mu_i}$$

$$\tau_i = \frac{1 - \epsilon_i}{\lambda_i}$$

Note that if  $|C_i| \geq 3$ ,

$$\mu_i < \frac{\sum_{v_j \in B_i} D_j}{|C_i| - 1} \leq \frac{\sum_{v_j \in B_i} D_j}{2},$$

so  $\lambda_i > 2$ . For  $S \subseteq V - R$ ,  $\eta > 0$ ,  $\mathcal{Y}_S(\eta) \subseteq \mathbb{R}_{>0}^n$  is the set of  $W \in \mathbb{R}_{>0}$  such that

$$\frac{\min_{i \in S} W_i}{\sum_{i \in [n] - S} W_i} > \eta$$

The following theorems give bounds on the size of the weights after executing Algorithm A, and the number of steps required to execute the algorithm.

**Theorem 21.** *For each component  $C_i$  with  $|C_i| \geq 3$ , after executing Algorithm A, there is still a vertex with weight 1, and the  $m$ th smallest weight for  $m \geq 2$  is at most*

$$\left( \frac{\lambda_i}{\epsilon_i} - 1 \right) \left( \frac{\lambda_i}{\epsilon_i} \right)^{m-2}$$

**Theorem 22.** *The number of steps Algorithm A requires for component  $C_i$  is*

$$O\left(|C_i|^2 \lambda_i \ln(\lambda_i)\right)$$

Note that if  $|C_i| = 1$ , the algorithm need not be run; the vertex is in  $R$  and so has a weight of 1. If  $C_i = 2$ , either both  $F_j$  functions are 0 when both weights are 1, or one is positive and one is negative. If  $R_i$  is chosen to be the positive one, the algorithm doesn't need to be run. The following lemma is needed to prove Theorem 21.

**Lemma 23.** *Let  $S \subsetneq C_i$ ,  $|C_i| \geq 3$ . If  $W \in \mathcal{Y}_S\left(\frac{\lambda_i}{\epsilon_i} - 1\right)$ , then there exists a vertex  $v_j \in S$  with*

$$F_j(W) < -(1 - \epsilon_i)\mu_i$$

*Proof.* By the pigeon-hole principle it suffices to show that

$$\sum_{v_j \in S} F_j(W) < -(1 - \epsilon_i)\mu_i |S|.$$

By Lemma 11

$$\begin{aligned}
\sum_{v_j \in S} F_j(W) &< -|S|\mu_i + \frac{1}{\frac{\lambda_i}{\epsilon_i} - 1 + 1} \sum_{v_k \in B_i} D_k \\
&= -|S|\mu_i + \frac{\epsilon_i \sum_{v_k \in B_i} D_k}{\lambda_i} = -|S|\mu_i + \epsilon_i \mu_i \\
&\leq -|S|\mu_i + |S|\epsilon_i \mu_i = -|S|(1 - \epsilon_i)\mu_i
\end{aligned}$$

□

*Proof of Theorem 21.* Since each component  $C_i$  has exactly one vertex in  $R$ , this vertex has weight 1. The bounds on the remaining weights are proven inductively using that as the base case. So suppose the first  $m - 1$  smallest weights have the given bound. The bound for the  $m$ th largest weight is proven by contradiction. Let  $S'$  be the set containing the  $m - 1$  smallest weights, so  $S = C_i - S'$  is the set containing all of the larger weights. The assumption for contradiction is

$$\min_{v_j \in S} W_j > \left( \frac{\lambda_i}{\epsilon_i} - 1 \right) \left( \frac{\lambda_i}{\epsilon_i} \right)^{m-2}.$$

By induction,

$$\begin{aligned}
\sum_{v_k \in S'} W_k &\leq 1 + \sum_{k=2}^{m-1} \left( \frac{\lambda_i}{\epsilon_i} - 1 \right) \left( \frac{\lambda_i}{\epsilon_i} \right)^{k-2} \\
&= 1 + \left( \frac{\lambda_i}{\epsilon_i} - 1 \right) \sum_{k=0}^{m-3} \left( \frac{\lambda_i}{\epsilon_i} \right)^k \\
&= 1 + \left( \frac{\lambda_i}{\epsilon_i} - 1 \right) \frac{1 - \left( \frac{\lambda_i}{\epsilon_i} \right)^{m-2}}{1 - \frac{\lambda_i}{\epsilon_i}} \\
&= \left( \frac{\lambda_i}{\epsilon_i} \right)^{m-2}
\end{aligned}$$

This implies that  $W \in \mathcal{Y}_S \left( \frac{\lambda_i}{\epsilon_i} - 1 \right)$ . By Lemma 23 there exists  $v_j \in S$  with  $F_j(W) < -(1 - \epsilon_i)\mu_i$ . But  $W_j > \left( \frac{\lambda_i}{\epsilon_i} - 1 \right) \left( \frac{\lambda_i}{\epsilon_i} \right)^{m-2} > 1$ , so its weight must have been increased by Algorithm A. By Lemma 12,  $F_j(W) > -(1 - \epsilon_i)\mu_i$ , which is a contradiction. □

Theorem 21 bounds the ending weights after Algorithm A terminates. Since at least one weight is multiplied by  $1 + \tau_i$  at each step of the algorithm, this can be used to bound the number of steps where particular weight is increased. Summing these bounds then provides a bound on the total number of steps the algorithm can take. In particular, the number of steps Algorithm A takes when executed on  $C_i$  is bounded by

$$\begin{aligned}
& \sum_{m=2}^{|C_i|} \frac{\ln \left( \left( \frac{\lambda_i}{\epsilon_i} - 1 \right) \left( \frac{\lambda_i}{\epsilon_i} \right)^{m-2} \right)}{\ln(1 + \tau_i)} \\
& \leq \sum_{m=1}^{|C_i|} \frac{m \ln \left( \frac{\lambda_i}{\epsilon_i} \right)}{\ln \left( 1 + \frac{1 - \epsilon_i}{\lambda_i} \right)} \\
& = \binom{|C_i| + 1}{2} \frac{\ln \left( \frac{\lambda_i}{\epsilon_i} \right)}{\ln \left( 1 + \frac{1 - \epsilon_i}{\lambda_i} \right)} \tag{2.5}
\end{aligned}$$

This is where  $\epsilon_i$  takes effect. Choosing  $\epsilon_i = 0$  pushes the numerator to  $\infty$ , whereas choosing  $\epsilon_i = 1$  pushes the denominator to 0. Neither of these are good choices, but there is a minimum somewhere in between. The following technical lemma will be used to bound the effect of  $\epsilon_i$ .

**Lemma 24.** *For  $0 < x \leq c$*

$$\frac{1}{x} + \frac{1}{\ln(1 + c)} - \frac{1}{c} \leq \frac{1}{\ln(1 + x)} \leq \frac{1}{x} + \frac{1}{2}$$

*Proof.* Note that it suffices to show that

$$\lim_{x \rightarrow 0} \frac{1}{\ln(1 + x)} - \frac{1}{x} = \frac{1}{2},$$

and that  $\frac{1}{\ln(1 + x)} - \frac{1}{x}$  is decreasing for  $x > 0$ . The first is a straightforward application of L'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{1}{\ln(1 + x)} - \frac{1}{x} = \lim_{x \rightarrow 0} \frac{x - \ln(1 + x)}{x \ln(x + 1)} = \lim_{x \rightarrow 0} \frac{1 - \frac{1}{x+1}}{\ln(x + 1) + \frac{x}{x+1}}$$

$$= \lim_{x \rightarrow 0} \frac{x}{(x+1)\ln(x+1) + x} = \lim_{x \rightarrow 0} \frac{1}{2 + \ln(x+1)} = \frac{1}{2}$$

To show that  $\frac{1}{\ln(1+x)} - \frac{1}{x}$  is decreasing, note that its derivative is

$$\frac{(x+1)\ln(x+1)^2 - x^2}{x^2(x+1)\ln(x+1)^2}$$

Since  $x > 0$ , it suffices to show that the numerator is negative.

$$\begin{aligned} & (x+1)\ln(x+1)^2 - x^2 \\ &= (\sqrt{x+1}\ln(x+1) + x)\sqrt{x+1} \left( \ln(x+1) - \sqrt{x+1} + \frac{1}{\sqrt{x+1}} \right) \end{aligned}$$

As this is 0 when  $x$  is 0, it suffices to show that the derivative of

$$\ln(x+1) - \sqrt{x+1} + \frac{1}{\sqrt{x+1}}$$

is negative. Its derivative is

$$\frac{1}{x+1} - \frac{1}{2(x+1)} \left( \sqrt{x+1} + \frac{1}{\sqrt{x+1}} \right) = \frac{1}{2(x+1)} \left( 2 - \sqrt{x+1} - \frac{1}{\sqrt{x+1}} \right),$$

thus it suffices to show

$$\sqrt{x+1} + \frac{1}{\sqrt{x+1}} \geq 2$$

They are equal at  $x = 0$ , and the left hand side is increasing.  $\square$

Using Lemma 24 and the fact that  $\lambda_i > 2$  when  $|C_i| \geq 3$ , Expression 2.5 can be bounded below by

$$\binom{|C_i|+1}{2} \ln(\lambda_i) \left( \lambda_i + \frac{1}{\ln\left(\frac{3}{2}\right)} - 2 \right) \leq \binom{|C_i|+1}{2} \frac{\ln\left(\frac{\lambda_i}{\epsilon_i}\right)}{\ln\left(1 + \frac{1-\epsilon_i}{\lambda_i}\right)}$$

Theorem 22 cannot be improved by a better choice of  $\epsilon_i$ .

*Proof of Theorem 22.* Using Lemma 24, Expression 2.5 can be bounded above by

$$\binom{|C_i|+1}{2} \frac{\ln\left(\frac{\lambda_i}{\epsilon_i}\right)}{\ln\left(1 + \frac{1-\epsilon_i}{\lambda_i}\right)} \leq \binom{|C_i|+1}{2} \ln\left(\frac{\lambda_i}{\epsilon_i}\right) \left( \frac{\lambda_i}{1-\epsilon_i} + \frac{1}{2} \right).$$

Fixing  $\epsilon_i = \frac{1}{2}$  turns this into

$$\begin{aligned} & 2 \binom{|C_i| + 1}{2} \ln(2\lambda_i) (\lambda_i + 1) \\ & = O(|C_i|^2 \lambda_i \ln(\lambda_i)). \end{aligned} \tag{2.6}$$

□

This asymptotic bound cannot be improved by different choices of  $\epsilon_i$ , but the upper bound given by Expression 2.6 can be improved upon by using  $\epsilon_i$  values other than  $1/2$ .

**Proposition 25.**

$$\min_{x \in (0,1)} \frac{1}{1-x} \ln\left(\frac{\lambda}{x}\right) \leq \ln(e\lambda) + \ln(\ln(e\lambda)) + \ln\left(1 + \frac{\ln(\ln(e\lambda \ln(4\lambda^2)))}{\ln(e\lambda)}\right)$$

The following lemma is proven first, to give a bound in the proof of Proposition 25.

**Lemma 26.** *For all  $n > 0$*

$$\ln(ex) \leq \frac{e^{n-1}x^n}{n}$$

*Proof.* First note that both sides are equal when  $x = e^{-\frac{n-1}{n}}$ . It suffices to show that  $\ln(ex) - \frac{e^{n-1}x^n}{n}$  has a negative derivative when  $x > e^{-\frac{n-1}{n}}$ , and a positive derivative when  $0 < x < e^{-\frac{n-1}{n}}$ . The derivative is

$$\frac{1}{x} - (ex)^{n-1} = \frac{1 - e^{n-1}x^n}{x}$$

Since  $n > 0$ , the numerator is negative when  $x > e^{-\frac{n-1}{n}}$ , and positive when  $x < e^{-\frac{n-1}{n}}$ .  $x > 0$ , so this completes the proof. □

*Proof of Proposition 25.* Let  $f(x) = \frac{1}{1-x} \ln\left(\frac{\lambda}{x}\right)$ .  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 1} f(x) = \infty$ .  $f(x)$  is minimized at a point  $x_0$  such that  $f'(x_0) = 0$

$$f'(x_0) = \frac{1}{(1-x_0)^2} \ln\left(\frac{\lambda}{x_0}\right) - \frac{1}{x_0(1-x_0)} = 0$$

$$\frac{1}{x_0} = \frac{1}{1-x_0} \ln\left(\frac{\lambda}{x_0}\right) = f(x_0)$$

Rearranging again,

$$\begin{aligned} \ln\left(\frac{\lambda}{x_0}\right) &= \frac{1-x_0}{x_0} = \frac{1}{x_0} - 1 \\ f(x_0) &= \frac{1}{x_0} = \ln\left(\frac{e\lambda}{x_0}\right) \end{aligned} \tag{2.7}$$

$$f(x_0) = \ln(e\lambda f(x_0)) \tag{2.8}$$

Applying Lemma 26 with  $n = \frac{1}{2}$  to Equation 2.7 implies

$$\frac{1}{x_0} \leq \frac{2\lambda^{\frac{1}{2}}}{(ex_0)^{\frac{1}{2}}}$$

$$\frac{1}{x_0^2} \leq \frac{4\lambda}{ex_0}$$

$$\frac{1}{x_0} \leq \frac{4\lambda}{e}$$

$$f(x_0) \leq \frac{4\lambda}{e}$$

Better bounds are obtained by repeatedly substituting the Equation 2.8 into itself.

$$\begin{aligned} f(x_0) &= \ln(e\lambda \ln(e\lambda f(x_0))) = \ln(e\lambda) + \ln(\ln(e\lambda \ln(e\lambda f(x_0)))) \\ &= \ln(e\lambda) + \ln(\ln(e\lambda)) + \ln\left(1 + \frac{\ln(\ln(e\lambda f(x_0)))}{\ln(e\lambda)}\right) \\ &= \ln(e\lambda) + \ln(\ln(e\lambda)) + \ln\left(1 + \frac{\ln(\ln(e\lambda \ln(e\lambda f(x_0))))}{\ln(e\lambda)}\right) \\ &\leq \ln(e\lambda) + \ln(\ln(e\lambda)) + \ln\left(1 + \frac{\ln(\ln(e\lambda \ln(4\lambda^2)))}{\ln(e\lambda)}\right) \end{aligned}$$

□

It is worth noting that we can actually solve equation 2.7 for  $x_0$  using the  $-1$  branch of the Lambert- $W$  function. The Lambert- $W$  function is the inverse of the function  $x \exp(x)$ .  $x \exp(x)$  is not one-to-one over  $\mathbb{R}$ , but restricted to either of the two domains  $x \leq -1$  or  $x \geq -1$ , it is.  $W_0(x)$  is the inverse of the  $x \geq -1$  branch, and  $W_{-1}(x)$  is the inverse of the  $x \leq -1$  branch.

From Equation 2.7,

$$\begin{aligned}\frac{1}{x_0} &= \ln\left(\frac{e\lambda}{x_0}\right) \\ x_0 \ln\left(\frac{x_0}{e\lambda}\right) &= -1 \\ \frac{x_0}{e\lambda} \ln\left(\frac{x_0}{e\lambda}\right) &= -\frac{1}{e\lambda} \\ \ln\left(\frac{x_0}{e\lambda}\right) \exp\left(\ln\left(\frac{x_0}{e\lambda}\right)\right) &= -\frac{1}{e\lambda}\end{aligned}$$

If  $\lambda > x_0$ ,  $\ln\left(\frac{x_0}{e\lambda}\right) < -1$ , so

$$\begin{aligned}\ln\left(\frac{x_0}{e\lambda}\right) &= W_{-1}\left(-\frac{1}{e\lambda_i}\right) \\ x_0 &= e\lambda \exp\left(W_{-1}\left(-\frac{1}{e\lambda}\right)\right) = \frac{-1}{W_{-1}\left(-\frac{1}{e\lambda}\right)} \\ f(x_0) &= -W_{-1}\left(\frac{-1}{e\lambda}\right)\end{aligned}$$

By choosing  $\epsilon_i = -W_{-1}^{-1}\left(-\frac{1}{e\lambda_i}\right)$ , the number of steps in Algorithm A is bounded by

$$\begin{aligned}\binom{|C_i|+1}{2} \ln\left(\frac{\lambda_i}{\epsilon_i}\right) \left(\frac{\lambda_i}{1-\epsilon_i} + \frac{1}{2}\right) &\leq \binom{|C_i|+1}{2} \left(-W_{-1}\left(\frac{-1}{e\lambda_i}\right)\right) \left(\lambda_i + \frac{1}{2}\right) \\ &\leq \binom{|C_i|+1}{2} \left(\lambda_i + \frac{1}{2}\right) \left(\ln(e\lambda_i) + \ln(\ln(e\lambda_i)) + \ln\left(1 + \frac{\ln(\ln(e\lambda_i \ln(4\lambda_i^2)))}{\ln(e\lambda_i)}\right)\right)\end{aligned}$$

It is worth noting that  $\epsilon_i = \frac{1}{\ln(e\lambda_i)}$  gives a similar bound without having to deal with the Lambert- $W$  function. When this  $\epsilon_i$  is chosen, the number of steps for Algorithm A can be bounded by

$$\begin{aligned}\binom{|C_i|+1}{2} \ln\left(\frac{\lambda_i}{\epsilon_i}\right) \left(\frac{\lambda_i}{1-\epsilon_i} + \frac{1}{2}\right) \\ &= \binom{|C_i|+1}{2} \left(\frac{\lambda_i \ln(e\lambda_i)}{\ln(\lambda_i)} + \frac{1}{2}\right) (\ln(\lambda_i) + \ln(\ln(e\lambda_i))) \\ &= \binom{|C_i|+1}{2} \left(\lambda_i \left(\ln(e\lambda_i) + \ln(\ln(e\lambda_i)) + \frac{\ln(\ln(e\lambda_i))}{\ln(\lambda_i)}\right) + \frac{(\ln(\lambda_i) + \ln(\ln(e\lambda_i)))}{2}\right),\end{aligned}$$

which is only slightly worse than previous bound.

## 2.5 CONVERGENCE RATE OF ALGORITHM B

A convergent sequence  $X(i) \rightarrow X$  converges with  $Q$ -order  $k > 0$  iff there exists  $C > 0$  such that

$$|X(i+1) - X| \leq C|X(i) - X|^k.$$

If a sequence converges with  $Q$ -order 2, it is said to converge  $Q$ -quadratically.

**Theorem 27.** *Let  $Y(i) \rightarrow \bar{Y}$  be the sequence produced by Algorithm B for  $\bar{F}$ .  $Y(i)$  converges to  $\bar{Y}$   $Q$ -quadratically.*

This is proven via a more general statement. A few definitions are needed to state it. Let  $G_{j,Y}$  be the  $j$ th coordinate of  $G_Y$ , with  $\nabla G_{j,Y}$  being its gradient, or equivalently the  $j$ th row of the Jacobian of  $G_Y$ . Then for  $\psi_j \in \mathbb{R}^n$ , with  $1 \leq j \leq n$ , define the matrix  $M(G_Y)(\psi)$ , whose  $j$ th row is  $\nabla G_{j,Y}(\psi_j)$ . Let  $\lambda_1(A)$  be the eigenvalue of matrix  $A$  with smallest absolute value.  $F'(X)$  is the Jacobian of  $F$  evaluated at  $X$ .  $[n]$  denotes the integers between and including 0 and  $n$ .  $\mathbb{N}_0$  is used to denote the non-negative integers. For  $L \in \mathbb{N}_0^n$ ,  $L = (L_1, \dots, L_n)$ ,  $|L| = \sum_{j=1}^n L_j$ ,  $X^L = X_1^{L_1} \dots X_n^{L_n}$ ,  $\frac{\partial^{|L|}}{\partial X^L} = \frac{\partial^{|L|}}{\partial X_1^{L_1} \dots \partial X_n^{L_n}}$ ,  $L! = L_1! \dots L_n!$ , and  $\binom{|L|}{L} = \frac{|L|!}{L!}$ .

**Theorem 28.** *Let  $Y(i) \in \mathbb{R}^n$  be a sequence converging to  $X$ , with  $F(X) = 0$ , for some function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with continuous  $(k+1)$ st derivatives. Let  $G_Z$  be a function such that  $G$  agrees with  $F$  on all derivatives of order  $k$  or less at  $Z$ . For any  $L \in \mathbb{N}_0^n$ ,  $|L| = k+1$ ,  $\partial^L G_Y(Z)$  is continuous in both  $Y$  and  $Z$  at  $X$ . Suppose  $G_{Y(i)}(Y(i+1)) = 0$  for all  $i$ . Let  $\mathcal{A}_i$  be the line segment between  $X$  and  $Y(i)$ . For  $L \in \mathbb{N}_0^n$ ,*

$$C = \max_{\substack{|L|=k+1 \\ \ell \in [n]}} \left| \frac{\partial^{k+1}(F_\ell - G_{\ell,X})(X)}{\partial X^L} \right|.$$

*Then there exists an integer  $N > 0$  such that for all  $i \geq N$ ,*

$$\min_{\psi \in \mathcal{A}_{i+1}^n} |\lambda_1(M(G_{Y(i)})(\psi))| |Y(i+1) - X| \leq \frac{2C\sqrt{n^{k+2}}}{(k+1)!} |Y(i) - X|^{k+1}$$

**Corollary 29.** *Suppose there exists  $\epsilon > 0$ , and a positive integer  $N$  such that for all  $i \geq N$ ,*

$$\min_{\psi \in \mathcal{A}^n} |\lambda_1(M(G_{Y(i)})(\psi))| > \epsilon.$$

*Then  $Y(i)$  converges to  $X$  with  $Q$ -order  $k + 1$ .*

**Corollary 30.** *Given the assumptions of Theorem 28, if  $\nabla G_{j,Z}(\psi_j)$  is continuous in both  $Z$  and  $\psi_j$  at  $Z = \psi_j = X$  for all  $j$ , and  $F'(X)$  is non-singular, then  $Y(i)$  converges with  $Q$ -order  $k + 1$ .*

*Proof of Corollary 30.* Let  $\epsilon = |\lambda_1(F'(X))| = |\lambda_1(G'_X(X))|$ . Since  $\nabla G_{j,Z}(\psi_j)$  is continuous in both  $Z$  and  $\psi_j$ , there exists a  $\delta$  so that if all of the  $\psi_j$  and  $Z$  are within  $\delta$  of  $X$ , then  $||\lambda_1(M(G'_Z)(\psi))| - \epsilon| < \frac{\epsilon}{2}$  is within  $\epsilon/2$  of  $\epsilon$ , and so  $|\lambda_1(M(G_Z)(\psi))| > \epsilon/2$ .  $Y(i)$  converges to  $X$ , so for large enough  $i$  it is within  $\delta$  of  $X$ , as is  $\psi_j$  for any  $\psi_j \in \mathcal{A}$ . By Corollary 29 this implies  $Y(i)$  converges with  $Q$ -order  $k + 1$ .  $\square$

*proof of Theorem 28.*

$$0 = F(X) = G_{Y(i)}(X) + (F - G_{Y(i)})(X) \tag{2.9}$$

$F$  and  $G_{Y(i)}$  agree on all derivatives of order  $k$  or less at  $Y(i)$ , so the Taylor expansion of  $F_j - G_{j,Y(i)}$  at  $Y(i)$  has no non-zero terms before the  $(k + 1)$ st derivative terms. There exists  $\xi_j$  on the line between  $X$  and  $Y(i)$  such that

$$(F_j - G_{j,Y(i)})(X) = \sum_{\substack{L \in \mathbb{N}_0^n \\ |L|=k+1}} \frac{\partial^{k+1}(F_j - G_{j,Y(i)})(\xi_j)}{\partial X^L} \frac{(X - Y(i))^L}{L!}$$

Let

$$C = \max_{\substack{|L|=k+1 \\ \ell \in [n]}} \left| \frac{\partial^{k+1}(F_\ell - G_{\ell,X})(X)}{\partial X^L} \right|.$$

By the continuity of the partial derivatives, for  $i$  large enough, all of the derivatives at  $\xi_j$  are bounded by  $2C$ .

$$|(F_j - G_{j,Y(i)})(X)| \leq 2C \sum_{|L|=k+1} \frac{|X - Y(i)|^L}{L!}$$

$$\begin{aligned}
&= \frac{2C}{(k+1)!} \sum_{|L|=k+1} \binom{k+1}{L} |X - Y(i)|^L = \frac{2C}{(k+1)!} \left( \sum_{\ell=1}^n |X_\ell - Y_\ell(i)| \right)^{k+1} \\
&\leq \frac{2C\sqrt{n^{k+1}}}{(k+1)!} |X - Y(i)|^{k+1} \\
&\quad |(F - G_{Y(i)})(X)| \leq \frac{2C\sqrt{n^{k+2}}}{(k+1)!} |X - Y(i)|^{k+1}
\end{aligned}$$

Returning to equation 2.9,

$$(F_j - G_{j,Y(i)})(X) = -G_{j,Y(i)}(X) = G_{j,Y(i)}(Y(i+1)) - G_{j,Y(i)}(X)$$

By the Mean Value Theorem, there exists  $\psi_j$  between  $X$  and  $Y(i+1)$  such that this is equivalent to:

$$(F_j - G_{j,Y(i)})(X) = \nabla G_{j,Y(i)}(\psi_j) \cdot (Y(i+1) - X).$$

Let  $M(G_{Y(i)})(\psi)$  be the matrix whose  $j$ th row is  $\nabla G_{j,Y(i)}(\psi_j)$ .

$$M(G_{Y(i)})(\psi) \cdot (Y_{i+1} - X)^T = (F - G_{Y(i)})(X)$$

$$\begin{aligned}
|\lambda_1(M(G_{Y(i)})(\psi))| |Y_{i+1} - X| &\leq |M(G_{Y(i)})(\psi) \cdot (Y_{i+1} - X)^T| \\
&= |(F - G_{Y(i)})(X)| \leq 2C\sqrt{n^{k+2}} |X - Y_i|^{k+1}
\end{aligned}$$

□

The following theorem, along with Corollary 17, prove that Corollary 30 applies to Algorithm B for  $\bar{F}$ , which proves Theorem 27

**Theorem 31.** *Suppose  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  has property  $\mathbb{P}$ . Then for  $X \in \mathbb{R}_{>0}^n$ ,  $F'(X)$  is non-singular.*

*Proof.*

$$F_i(X) = A_i - \sum_{j=1}^{m_i} \frac{X_i}{\sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i}$$

For  $i \neq \ell$ ,

$$\frac{\partial F_i}{\partial X_\ell}(X) = \sum_{j=1}^{m_i} \frac{\alpha_{j,\ell}^i X_i}{\left( \sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i \right)^2},$$

otherwise

$$\begin{aligned} \frac{\partial F_i}{\partial X_i}(X) &= \sum_{j=1}^{m_i} \frac{\alpha_{j,i}^i X_i}{\left( \sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i \right)^2} - \sum_{j=1}^{m_i} \frac{1}{\sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i} \\ &= - \sum_{j=1}^{m_i} \frac{\sum_{k \neq i}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i}{\left( \sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i \right)^2} \end{aligned}$$

Consider the matrix  $A = \text{diag}(X)^{-1} F'(X) \text{diag}(X)$ . For  $i \neq j$ ,

$$\begin{aligned} A_{i,\ell} &= \sum_{j=1}^{m_i} \frac{\alpha_{j,\ell}^i X_\ell}{\left( \sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i \right)^2} \\ A_{i,i} &= - \sum_{j=1}^{m_i} \frac{\sum_{k \neq i}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i}{\left( \sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i \right)^2} \\ \sum_{\ell=1}^n A_{i,\ell} &= - \sum_{j=1}^{m_i} \frac{\alpha_{j,n+1}^i}{\left( \sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i \right)^2}. \end{aligned}$$

Define the  $(n+1) \times (n+1)$  matrix  $B$  so that if  $1 \leq i, \ell \leq n$ ,  $B_{i,\ell} = A_{i,\ell}$ ,  $B_{n+1,\ell} = \frac{1}{n}$ ,  $B_{n+1,n+1} = -1$ , and

$$B_{i,n+1} = \sum_{j=1}^{m_i} \frac{\alpha_{j,n+1}^i}{\left( \sum_{k=1}^n \alpha_{j,k}^i X_k + \alpha_{j,n+1}^i \right)^2}.$$

The rows of  $B$  sum to 0, and only the diagonal elements are negative. Let  $b$  be the largest of the diagonal entries.  $bI + B$  has only non-negative entries, every row sums to  $b$ , and since  $\mathbb{G}_F$  has paths from every vertex  $\mathbb{V}_i$  to the vertex  $\mathbb{V}_{n+1}$ , by the property

$\mathbb{P}$ ,  $bI + B$  is irreducible. The Perron-Frobenius theorem implies that  $(bI + B)^T$  has  $b$  as an eigenvalue with algebraic multiplicity 1, and an eigenvector  $V$  with all positive coordinates.  $B^T V = 0$ , and  $B$  has rank  $n$ . The  $(n + 1)$ st row of  $B$  can be written as a linear combination of the other rows, and the  $(n + 1)$ st column can be written as a linear combination of the other columns, so their removal does not reduce the rank.  $A$  has rank  $n$ , so it is non-singular, and thus  $F'(X)$  is also non-singular.  $\square$

By the Inverse Function Theorem [23], Theorem 31 implies that  $\bar{F}$  is invertible in some open neighborhood for every  $X \in \mathbb{R}_{>0}^{n-c}$ . This means that there are at most countably many sets of weights which can give a particular set of expected occupation times.

## 2.6 SPECIAL CASES

This section discusses some special cases of the problem for which a little more can be said than in the general case. Specifically, it considers what can be said about simple graphs and some special cases thereof. Note that the graph  $G$  cannot be simple, as the end vertex has no out edges. However, a simple  $G$  may be presented, removing the offending entries in the adjacency matrix for calculations. For the remainder of the section  $G$  is a connected simple graph.

The process of finding the expected number of visits from the weights is relatively unchanged upon restricting to simple graphs, with the aforementioned exception. For the reverse direction however a few facts can be refined.

**Theorem 32.**  *$H$  has two components iff  $G$  is bipartite, and one component otherwise.*

The following lemma and corollary are useful in the proof.

**Lemma 33.**  *$v_i$  and  $v_j$  are neighbors in  $H$  iff there exists a path of length 2 between them in  $G$ .*

*Proof.* By the definition of  $H$ ,  $v_i$  and  $v_j$  are neighbors iff there exists  $v_k$  such that  $v_k$  has an out edge to each of  $v_i$  and  $v_j$ . Since  $G$  is simple this occurs iff  $v_i, v_k, v_j$  forms a path.  $\square$

**Corollary 34.**  *$v_i$  and  $v_j$  are in the same component in  $H$  iff there exists an even length walk between them in  $G$ .*

*proof of Theorem 32.* A bipartite  $G$  must give  $H$  two components as only vertices in the same partite set of  $G$  will have walks of even length between them. To see that any other  $G$  must give  $H$  one component, note that  $G$  contains an odd cycle. Let  $v_k$  be some vertex on that cycle. For any  $v_i$  and  $v_j$ , there is a walk from  $v_i$  to  $v_j$  which contains  $v_k$ . If this walk is even, then  $v_i$  and  $v_j$  are in the same component. If it is odd, then insert that odd cycle into the walk when it visits  $v_k$  to create an even length walk. Any two vertices are in the same component, so there is only one component.  $\square$

**Theorem 35.** *Suppose  $G$  is a path with  $n$  vertices from the start vertex to the end vertex. Let  $D_i$  be the expected number of visits to vertex  $v_i$ ,  $E_i = 1$  if  $i$  is odd, and  $E_i = 0$  if  $i$  is even. For  $1 \leq i \leq n - 1$ , the weights are given by*

$$W_i = \prod_{j=1}^{\lfloor \frac{i-1}{2} \rfloor} \left( \frac{D_{i+1-2j}}{\sum_{k=1}^{i-2j} (-1)^{k+i-2j} D_k + E_i} - 1 \right),$$

where the empty product is taken to be 1.  $W_n = W_{n-2} \left( \frac{D_{n-1}}{D_{n-2}} - 1 \right)$ .

*Proof.* Define  $P_{i,i+1} = \frac{W_{i+1}}{W_{i+1} + W_{i-1}}$ , and  $P_{i,i-1} = 1 - P_{i,i+1}$ . It suffices to show that  $D_{i-1}P_{i-1,i} + D_{i+1}P_{i+1,i} = D_i$  for  $2 \leq i \leq n - 2$ , as well as  $1 + D_2P_{2,1} = D_1$ ,  $D_{n-2}P_{n-2,n-1} = D_{n-1}$ , and  $D_{n-1}P_{n-1,n} = 1$ .

For  $2 \leq i \leq n-1$ ,

$$\begin{aligned}
\frac{W_{i+1}}{W_{i-1}} &= \frac{\prod_{j=1}^{\lfloor \frac{i}{2} \rfloor} \left( \frac{D_{i+2-2j}}{\sum_{k=1}^{i+1-2j} (-1)^{k+i+1-2j} D_k + E_{i+1}} - 1 \right)}{\prod_{j=1}^{\lfloor \frac{i-2}{2} \rfloor} \left( \frac{D_{i-2j}}{\sum_{k=1}^{i-1-2j} (-1)^{k+i-1-2j} D_k + E_{i-1}} - 1 \right)} \\
&= \frac{D_i}{\sum_{k=1}^{i-1} (-1)^{k+i-1} D_k + E_{i+1}} - 1 \\
P_{i,i+1} &= \frac{W_{i+1}}{W_{i+1} + W_{i-1}} \\
&= \frac{W_{i+1}/W_{i-1}}{W_{i+1}/W_{i-1} + 1} \\
&= \frac{\left( \frac{D_i}{\sum_{k=1}^{i-1} (-1)^{k+i-1} D_k + E_{i+1}} - 1 \right)}{\left( \frac{D_i}{\sum_{k=1}^{i-1} (-1)^{k+i-1} D_k + E_{i+1}} - 1 + 1 \right)} \\
&= \frac{\sum_{k=1}^i (-1)^{k+i} D_k + E_{i+1}}{D_i} \\
P_{i,i-1} &= \frac{\sum_{k=1}^{i-1} (-1)^{k+i-1} D_k - E_{i-1}}{D_i}
\end{aligned}$$

$$1 + D_2 P_{2,1} = 1 + D_1 - 1 = D_1$$

$$D_{i-1} P_{i-1,i} + D_{i+1} P_{i+1,i} = \sum_{k=1}^{i-1} (-1)^{k+i-1} D_k + E_i + \sum_{k=1}^i (-1)^{k+i} D_k - E_i = D_i.$$

$G$  is bipartite, so  $H$  has two components,  $C_1$  and  $C_2$ , splitting the odd and even vertices. Without loss of generality, assume  $v_n \in C_1$ .  $\xi_{C_1} = E_n$ . By Theorem 3,

$$\begin{aligned}
\sum_{v_i \in C_1} D_j &= \sum_{v_j \in N^-(C_1)} D_j + E_n. \\
\sum_{k=1}^n (-1)^{k+n} D_k &= E_n. \\
D_{n-2} P_{n-2, n-1} &= \sum_{k=1}^{n-2} (-1)^{k+n-2} D_k + E_{n-1} \\
&= D_{n-1} - 1 + \sum_{k=1}^n (-1)^{k+n} D_k + E_{n-1} = D_{n-1} + E_n + E_{n-1} - 1 \\
&= D_{n-1} \\
D_{n-1} P_{n-1, n} &= \sum_{k=1}^{n-1} (-1)^{k+n-1} D_k + E_n = 1 - \sum_{k=1}^n (-1)^{k+n} D_k + E_n \\
&= 1 - E_n + E_n = 1
\end{aligned}$$

□

**Corollary 36.** *The weights given by Theorem 35 are the unique solution for  $D$  on a path with  $D_1 = D_2 = 1$ .  $P$  is uniquely determined by  $D$ .*

*Proof.* If the first  $i$  weights are determined uniquely, then the equation  $D_{i-1} P_{i-1, i} + D_{i+1} P_{i+1, i} = D_i$  has a unique solution for  $P_{i+1, i}$ , so there is a unique solution for  $W_{i+1}$ . □

Recall  $P(W)$  is the matrix of probabilities for the weights  $W$ . Let  $D(W)$  be the expected occupation times for  $W$ . For an induced subgraph  $G'$  of  $G$ ,  $W|_{G'}$  is the restriction of  $W$  to vertices in  $G'$ . A leaf in  $G$  is a vertex with only one neighbor.

**Theorem 37.** *Let  $G$  be a graph with at least 3 vertices, one of which,  $x$ , is a leaf,  $x \neq v_1$ ,  $x \neq v_n$ , and  $x$  is not adjacent to  $v_n$ .  $G' = G - x$ . Let  $W_1$  and  $W_2$  be weights on the vertices of  $G$ . If  $D(W_1) = D(W_2)$  then  $D(W_1|_{G'}) = D(W_2|_{G'})$ . If  $P(W_1) \neq P(W_2)$  then  $P(W_1|_{G'}) \neq P(W_2|_{G'})$ .*

*Proof.* Let  $P_1 = P(W_1)$ ,  $P_2 = P(W_2)$ ,  $D = D(W_1)$ .  $P_1 D = D = P_2 D$ .  $x$  has only one neighbor, call it  $y$ .  $D_x$  is equal to  $D_y$  times the probability of going from  $y$  to  $x$ . This probability must be the same for both  $P_1$  and  $P_2$ , call it  $p$ . Let  $\bar{P}_1$  be the matrix  $P_1$  with row and column  $x$  removed, and let  $K$  be the identity matrix with the  $y$  entry replaced by  $1/(1-p)$ .  $P'_1 = P(W_1|_{G'}) = \bar{P}_1 K$ . Let  $\bar{D}$  be  $D$  with the  $x$  entry removed.  $\bar{P}_1 \bar{D} = K^{-1} \bar{D}$ .

$$P'_1 K^{-1} \bar{D} = \bar{P}_1 \bar{D} = K^{-1} \bar{D}$$

$$K^{-1} \bar{D} = D(W_1|_{G'})$$

$$P'_2 = P(W_2|_{G'}) = \bar{P}_2 K$$

$$K^{-1} \bar{D} = D(W_1|_{G'})$$

It suffices to show that  $P_1$  and  $P_2$  differ in some entry other than the probability of going from  $y$  to  $x$ , or from  $x$  to  $y$ . There are at least 3 vertices,  $x$  is not the start or end vertex, and  $y$  is not the end vertex. There is a walk from  $y$  to  $v_n$ , so  $y$  must have another neighbor,  $z$ . Without loss of generality, assume that  $z$  has the same weight in both  $W_1$  and  $W_2$ . If the weights of the neighbors of  $y$  have different sums in  $W_1$  versus  $W_2$ , then the probability of going from  $y$  to  $z$  differs between  $P_1$  and  $P_2$ . So now suppose the sums are the same. If  $x$  also has the same weight in both, then they must differ somewhere else as they are not equal. If  $x$  has different weights in  $W_1$  versus  $W_2$ , then some neighbor of  $y$  must have different weights for the sums to be equal. The probability of going from  $y$  to this vertex differs between  $P_1$  and  $P_2$ .  $\square$

**Corollary 38.** *If  $P$  is uniquely determined by  $D$  on  $G$  for any  $D$  satisfying Theorem 3, then the same is true of  $G$  with a leaf added to any vertex other than  $v_n$ .*

Thus combining Corollaries 36 and 38, we get the following

**Corollary 39.** *If  $G$  is a tree, then  $P$  is uniquely determined by  $D$ .*

# CHAPTER 3

## GRAPHICAL PROJECTIVE SPACES

### 3.1 INTRODUCTION AND DEFINITIONS

The primary motivating force behind the definition of graphical projective spaces is as an appropriate extension of the space from which weights are pulled for the vertices of a vertex-weighted graph. In particular they will be used to complete the space of possible solutions for the random walk problem discussed in Chapter 2 on a vertex-weighted graph. However, they also provide an interesting connection between hypergraphs and the topological spaces which arise out of the process, which may hold independent interest.

Given a hypergraph  $H$  on  $n$  vertices, a function will be defined from the first orthant of  $\mathbb{R}^n$  to the interior of the unit cube in an appropriate  $\mathbb{R}^m$ . This will induce an equivalence relation on the initial orthant, and a metric on the resulting quotient space. The completion of the quotient space with respect to the metric can then be formed. The resulting space is a graphical projective space.

More formally, let  $H$  be a hypergraph with vertex set  $V$  and edge set  $E$ , composed of  $c$  components. For  $v \in V$ , define the degree  $d(v)$  as the number of edges  $e \in E$  with  $v$  incident to  $e$ . Set  $n = |V|$  and  $m = \sum_{v \in V} d(v)$ . Define  $f : \mathbb{R}_{>0}^n \rightarrow (0, 1)^m$ .  $f$  is described as a collection of  $m$  functions from  $\mathbb{R}_{>0}$  to  $(0, 1)$ , it has a separate coordinate function for each incident vertex-edge pair. The input coordinates are

indexed by the vertices of  $V$ . For  $v_i \in e \in E$

$$f_{e,v_i}(X) = \frac{X_i}{\sum_{v_j \in e} X_j}$$

More intuitively, the input is a collection of finite positive weights on the vertices, and for each vertex-edge pair the function returns the proportion of that edge's weight which is at that vertex.

$f$  provides an equivalence relation on  $\mathbb{R}_{>0}^n$  of  $X \approx Y$  iff  $f(X) = f(Y)$ . Each equivalence class contains a unique element, minimal in the lexicographic order on  $\mathbb{R}_{>0}^n$ , with  $\min_i X_i = 1$ . Whenever an element of  $\mathbb{R}_{>0}^n / \approx$  is used it is assumed to be this minimal representative of the class. If  $H$  consists of the single hyperedge containing all vertices, then the equivalence relation noted is precisely the equivalence relation used in defining projective space, restricted to the open first orthant.

Define  $\tilde{f} : \mathbb{R}_{>0}^n / \approx \rightarrow (0, 1)^m$ , and define the metric  $d$  on  $\mathbb{R}_{>0}^n / \approx$  as

$$d(X, Y) = |\tilde{f}(X) - \tilde{f}(Y)|,$$

where  $|\cdot|$  is the  $L_2$  norm. Under this metric there are Cauchy sequences in  $\mathbb{R}_{>0}^n / \approx$  which do not converge inside the space, sequences where some coordinates head off towards  $\infty$ . Taking the completion of  $\mathbb{R}_{>0}^n / \approx$  with respect to the metric  $d$  adds points which essentially capture the possibilities for when the weights on the hypergraph head towards 0 or  $\infty$ . From here on the completed space will be called  $P_H$ . For  $X \in P_H$ , define  $\mathcal{S}(X)$  to be the set of vertices whose coordinates are unbounded in all Cauchy representatives of  $X$ . In the case where  $H$  is a single hyperedge with all of the vertices, the completion is the closed first orthant of projective space.  $\tilde{f}$  is extended to  $\bar{f} : P_H \rightarrow [0, 1]^m$ .  $\bar{f}$  is a homeomorphism between  $P_H$  and its image.

A topological space  $X$  is a Hausdorff space if for any  $x, y \in X$ ,  $x \neq y$ , there exist neighborhoods of  $x$  and  $y$  that are disjoint.  $\mathbb{R}^n$  is Hausdorff, and the property of being Hausdorff is preserved under homeomorphisms, and subspaces.  $[0, 1]^m$  is

Hausdorff, so  $P_H$  is Hausdorff. An  $n$ -dimensional open cell is a topological space which is homeomorphic to the open ball  $B_n$ . A Hausdorff space  $X$  along with a partition into open cells  $\mathcal{E}$  is a CW-complex if the following properties hold:

1. For every open  $n$ -cell  $C \in \mathcal{E}$ , there is a continuous map  $g_C : \overline{B}_n \rightarrow X$  such that  $g_C$  restricted to the interior  $B_n$  is a homeomorphism with  $C$ , and  $g_C$  maps any point on the boundary of  $\overline{B}_n$  into a cell in  $\mathcal{E}$  of dimension less than  $n$ .
2. For every  $n$ -cell  $C \in \mathcal{E}$ , the closure in  $X$ ,  $\overline{C}$ , intersects only finitely many other cells in  $\mathcal{E}$ .
3. A subset  $A \subseteq X$  is closed iff  $A \cap \overline{C}$  is closed for every  $C \in \mathcal{E}$ .

The last two properties are automatically satisfied if there are only finitely many cells in the partition.

**Proposition 40.** *Suppose the interior of  $X$  is an open  $n$ -cell, and there is a continuous function  $g : \overline{B}_n \rightarrow X$  such that the restriction of  $g$  to the open  $n$ -ball is a homeomorphism with the interior of  $X$ . If the boundary of  $X$  is a CW-complex with finitely many cells  $\mathcal{E}$ , each of dimension less than  $n$ , then  $X$  is a CW-complex with a cell decomposition formed by adding the interior of  $X$  to  $\mathcal{E}$ .*

*Proof.* All of the cells on the boundary of  $X$  satisfy the requirements as they are already part of a CW-complex. So it suffices to show that the cell formed by the interior does as well.  $g$  provides the requisite function, and the restrictions on the CW-complex formed by the boundary ensure that the image of the boundary of  $\overline{B}_n$  under  $g$  is contained in cells from  $\mathcal{E}$ , all of which have dimension less than  $n$ .  $\square$

**Proposition 41.** *Suppose  $X$  is a Hausdorff space with a partition into open cells  $\mathcal{E}$ . Suppose there is a finite collection  $\mathcal{D}$  of closed cells, each  $D \in \mathcal{D}$  is the union of open cells in  $\mathcal{E}$ . Each open cell  $C \in \mathcal{E}$  is in at least one  $D$ . If every  $D$  is a CW-complex over the open cells it contains, then  $X$  is a CW-complex.*

*Proof.* Since each open cell of  $X$  is contained in a CW-complex,  $X$  inherits the local properties for that cell. The third property is satisfied in each of the closed cells in  $\mathcal{D}$ , and there are only finitely many of them, so  $X$  satisfies the third property.  $\square$

Let  $S$  be a decreasing chain of subsets of  $V$ , that is  $V = S_0 \supseteq S_1 \supseteq \dots \supseteq S_k$ . The chain induced hypergraph  $H \searrow S$  is a hypergraph with the same vertices  $V$  as  $H$ ,  $e \searrow S$  is an edge in  $H \searrow S$  iff there exists an edge  $e \in E$  and an index  $i \in [k]$  such that  $e \searrow S = e \cap S_i \neq \emptyset$ , and for all  $j > i$ ,  $e \cap S_j = \emptyset$ .

### 3.2 CW-COMPLEXES ON GRAPHICAL PROJECTIVE SPACES

Define an equivalence relation on  $X, Y \in P_H$ ,  $X \sim Y$  iff there exists  $K \in \mathbb{R}_{>0}^n$  such that, for any Cauchy representative of  $X$ ,  $A$ , the sequence  $(K_1 A_1(i), \dots, K_n A_n(i))$  is a representative of  $Y$ . This is obviously reflective, and easily seen to be symmetric and transitive, so it is an equivalence relation. Let  $\mathcal{E}_H$  be the equivalence classes formed by  $\sim$ .  $\mathcal{E}_H$  is the collection of cells for the CW-complex of  $P_H$ . Note that for any  $C \in \mathcal{E}_H$ , if  $X, Y \in C$ , then  $\mathcal{S}(X) = \mathcal{S}(Y)$ , so the set  $\mathcal{S}(C)$  is well defined.

The equivalence class which contains  $A(i) = (1, 1, 1, \dots, 1)$  is the initial quotient space before the completion, so it is the interior of  $P_H$ , which is represented by  $\mathbb{R}_{>0}^n / \approx$ . This is homeomorphic to  $\mathbb{R}_{>0}^{n-c}$ , and therefore to the open ball  $B_{n-c}$ . There exists a continuous map  $h : \overline{B}_{n-c} \rightarrow P_H$  which restricts to a homeomorphism between the open ball  $B_{n-c}$  and the interior of  $P_H$ . Proposition 40 applies to  $P_H$ , so by Proposition 41 it suffices to find a collection  $\mathcal{D}$  of closed cells on the boundary so that every open cell in  $\mathcal{E}_H$  is contained in at least one  $D \in \mathcal{D}$ , and to show that  $\mathcal{D}$  forms a CW-complex.

Pick a nonempty subset  $S_1 \subsetneq V$ , such that every vertex in  $S_1$  is connected to some vertex in  $V - S_1$ . Let  $S$  be the chain  $V = S_0 \supseteq S_1$ .  $\mathcal{V}_{S_1}$  is the vector with a 1 for every vertex in  $S_1$ , and a 0 otherwise.  $Q_S$  is the the equivalence class containing

the point with the Cauchy sequence representative  $A(i) = (1, 1, \dots, 1) + i\mathcal{V}_{S_1}$ .  $\overline{Q_S}$  is the closure of  $Q_S$  in  $P_H$ .  $\mathcal{D}$  is the collection of these  $\overline{Q_S}$ .

**Lemma 42.** *Let  $S = \{V, S_1\}$ , with every vertex in  $S_1$  connected to  $V - S_1$ .  $X \in \overline{Q_S}$  iff for every edge  $e \in E$ , if  $e \cap S_1 \neq \emptyset$ , then for any  $v_i \in e - S_1$ ,  $\bar{f}_{e,v_i}(X) = 0$ .*

*Proof.* First suppose there exists an edge  $e \in E$  with  $e \cap S \neq \emptyset$ , and  $v_i \in e - S_1$  with  $\bar{f}_{e,v_i}(X) > 0$ . Pick  $Y \in Q_S$ . Let  $a(i)$  be a Cauchy sequence converging to  $Y$ . Without loss of generality, assume  $a_j(i) = a_j$  if  $v_j \notin S_1$ , and  $a_j(i) = a_j \cdot i$  if  $v_j \in S_1$ .

$$\bar{f}_{e,v_i}(Y) = \lim_{j \rightarrow \infty} \frac{a_i(j)}{\sum_{v_k \in e} a_k(j)} = \lim_{j \rightarrow \infty} \frac{a_i}{\sum_{v_k \in e \cap S_1} a_k \cdot j + \sum_{v_k \in e - S_1} a_k} = 0$$

$$d(X, Y) \geq \bar{f}_{e,v_i}(X) > 0$$

The distance from  $Y$  to  $X$  is bounded away from 0 for all  $Y \in Q_S$ , so  $X$  is not in the closure.

Now suppose for every edge  $e \in E$ , if  $e \cap S_1 \neq \emptyset$ , then for any  $v_i \in e - S_1$ ,  $\bar{f}_{e,v_i}(X) = 0$ . Let  $a$  be a representative Cauchy sequence for  $X$ , with  $a(i) = (a_1(i), a_2(i), \dots, a_n(i))$ . If  $e \cap S_1 \neq \emptyset$  and  $v_i \in e \cap S_1$ ,

$$\bar{f}_{e,v_i}(X) = \lim_{j \rightarrow \infty} \frac{a_i(j)}{\sum_{v_k \in e} a_k(j)} \leq \lim_{j \rightarrow \infty} \frac{a_i(j)}{\sum_{v_k \in e \cap S} a_k(j)}$$

$$1 = \lim_{j \rightarrow \infty} \sum_{v_i \in e} \frac{a_i(j)}{\sum_{v_k \in e} a_k(j)} = \sum_{v_i \in e} \bar{f}_{e,v_i}(X) = \sum_{v_i \in e \cap S_1} \bar{f}_{e,v_i}(X)$$

$$= \lim_{j \rightarrow \infty} \sum_{v_i \in e \cap S_1} \frac{a_i(j)}{\sum_{v_k \in e} a_k(j)} \leq \lim_{j \rightarrow \infty} \sum_{v_i \in e \cap S_1} \frac{a_i(j)}{\sum_{v_k \in e \cap S_1} a_k(j)} = 1$$

The inequality must actually be equality, so if  $e \cap S_1 \neq \emptyset$  and  $v_i \in e \cap S_1$ ,

$$\bar{f}_{e,v_i}(X) = \lim_{j \rightarrow \infty} \frac{a_i(j)}{\sum_{v_k \in e \cap S} a_k(j)}$$

For  $i \geq 1$ , let  $Y(i) \in Q_S$  be the element with representative Cauchy sequence  $b^i(k)$ , where  $b_j^i(k) = a_j(i)$  if  $v_j \notin S_1$ , and  $b_j^i(k) = k \cdot a_j(i)$  if  $v_j \in S_1$ . If  $e \cap S_1 = \emptyset$ ,

$\bar{f}_{e,v_j}(Y(i)) = \tilde{f}_{e,v_j}(a(i))$  If  $e \cap S_1 \neq \emptyset$ , and  $v_j \in e - S_1$ ,  $\bar{f}_{e,v_j}(Y(i)) = 0 = \bar{f}_{e,v_j}(X)$ . If  $e \cap S_1 \neq \emptyset$ , and  $v_j \in e \cap S_1$ ,

$$\begin{aligned} \bar{f}_{e,v_j}(Y(i)) &= \lim_{k \rightarrow \infty} \frac{b_j^i(k)}{\sum_{v_h \in e} b_h^i(k)} = \lim_{k \rightarrow \infty} \frac{k \cdot a_j(i)}{k \cdot \sum_{v_h \in e \cap S_1} a_h(i) + \sum_{v_h \in e - S_1} a_h(i)} \\ &= \frac{a_j(i)}{\sum_{v_h \in e \cap S_1} a_h(i)} \end{aligned}$$

$$\lim_{i \rightarrow \infty} d(Y(i), X) = 0$$

□

**Lemma 43.** For every cell  $C \in \mathcal{E}_H$ , and  $S = \{V, S_1\}$ , with every vertex in  $S_1$  connected to  $V - S_1$ , if  $C \cap \bar{Q}_S \neq \emptyset$ , then  $C \subset \bar{Q}_S$ .

*Proof.* Pick  $X \in C \cap \bar{Q}_S$ ,  $a(i)$  is a Cauchy sequence converging to  $X$ . For any other  $Y \in C$ , there exists  $K$  such that  $(K_1 a_1(i), \dots, K_n a_n(i))$  is a Cauchy sequence converging to  $Y$ .  $Y$  satisfies the conditions of Lemma 42. □

**Theorem 44.** Let  $C \in \mathcal{E}_H$ , and  $S = \{V, \mathcal{S}(C)\}$ .  $C \subset \bar{Q}_S$ .

*Proof.* Pick  $X \in C$ , with  $a(i)$  a Cauchy sequence converging to  $X$ . Every vertex in  $\mathcal{S}(C)$  is connected to a vertex in  $V - \mathcal{S}(C)$ . Let  $e \in E$  be given such that  $e \cap \mathcal{S}(C) \neq \emptyset$ , and choose  $v_i \in e - \mathcal{S}(C)$ . Since  $v_i \notin \mathcal{S}(C)$ , assume without loss of generality that  $a_i(j)$  is bounded as  $j$  goes to  $\infty$ .

$$\bar{f}_{e,v_i}(X) = \lim_{j \rightarrow \infty} \frac{a_i(j)}{\sum_{v_k \in e} a_k(j)} = \lim_{j \rightarrow \infty} \frac{a_i(j)}{\sum_{v_k \in e \cap \mathcal{S}(C)} a_k(j) + \sum_{v_k \in e - \mathcal{S}(C)} a_k(j)} = 0$$

By Lemmas 42 and 43,  $C \subset \bar{Q}_S$ . □

**Lemma 45.** If  $S = \{V, S_1\}$ , with every vertex in  $S_1$  connected to  $V - S_1$ , then  $\bar{Q}_S$  is isomorphic to  $P_{H \setminus S}$  as CW-complexes.

*Proof.* For  $X, Y \in \mathbb{R}_{>0}^n$ ,  $X \approx_H Y$  implies  $X \approx_{H \setminus S} Y$ , so the inclusion map  $\mathcal{I} : \mathbb{R}_{>0}^n / \approx_H \rightarrow \mathbb{R}^n / \approx_{H \setminus S}$  is well-defined. Pick  $X \in \overline{Q}_S$ . Let  $a(i)$  be a Cauchy sequence which converges to  $X$ . If  $e \subset S_1$  or  $e \subset V - S_1$ , then  $e = e \setminus S$ ,

$$\tilde{f}_{e, v_j}(a(i)) = \tilde{f}_{e \setminus S, v_j}(\mathcal{I}(a(i)))$$

If  $e \cap S_1 \neq \emptyset$  and  $v_j \in e - S_1$ ,

$$\bar{f}_{e, v_j}(X) = 0$$

If  $v_j \in e \cap S_1$ ,

$$\bar{f}_{e, v_j}(X) = \lim_{i \rightarrow \infty} \frac{a_j(i)}{\sum_{v_k \in e \cap S_1} a_k(i)} = \lim_{i \rightarrow \infty} \tilde{f}_{e \setminus S, v_j}(\mathcal{I}(a(i)))$$

$\mathcal{I}(a(i))$  is Cauchy in  $P_{H \setminus S}$ . Define  $F : \overline{Q}_S \rightarrow P_{H \setminus S}$  as  $F(X) = \lim_{i \rightarrow \infty} \mathcal{I}(a(i))$ .  $\bar{f}_{e \setminus S, v_j}(F(X)) = \bar{f}_{e, v_j}(X)$ , so  $F$  is one-to-one.  $\mathcal{I}$  is continuous, so  $F$  is as well. If  $X$  and  $Y$  are in the same cell, then  $F(X)$  and  $F(Y)$  will be too, and if  $X$  and  $Y$  are in different cells, then  $F(X)$  and  $F(Y)$  are as well. It suffices to show that  $F$  is onto.

Pick  $Y \in P_{H \setminus S}$ , and let  $b(i)$  be a Cauchy sequence in the quotient space of  $\mathbb{R}_{>0}^n$  over the equivalence relation  $\approx_{H \setminus S}$  which converges to  $Y$ . Let  $\beta(i) \in \mathbb{R}_{>0}^n$  be a sequence which maps to  $b(i)$ . Assume without loss of generality that  $\beta_j(i) \geq 1$ .  $K(i) = \sum_{v_j \in V - S_1} \beta_j(i)$ . Define  $\alpha(i)$  so that  $\alpha_j(i) = \beta_j(i)$  if  $v_j \notin S_1$ , and  $\alpha_j(i) = i \cdot \beta_j(i) \cdot (1 + K(i))$  if  $v_j \in S_1$ . Since  $S_1$  is the union of components in  $H \setminus S$ ,  $\alpha(i)$  is also a representative sequence for  $b(i)$ , and therefore for  $Y$ . Suppose  $e \subset S_1$  or  $e \subset V - S_1$ .

$$\lim_{i \rightarrow \infty} f_{e, v_j}(\alpha(i)) = \lim_{i \rightarrow \infty} \frac{\beta_j(i)}{\sum_{k \in e} \beta_k(i)} = \lim_{i \rightarrow \infty} f_{e \setminus S, v_j}(\beta(i)) = \bar{f}_{e \setminus S}(Y)$$

If  $e \cap S_1 \neq \emptyset$  and  $v_j \in e - S_1$ ,

$$\lim_{i \rightarrow \infty} f_{e, v_j}(\alpha(i)) = \lim_{i \rightarrow \infty} \frac{\beta_j(i)}{\sum_{v_k \in e \cap S_1} i \cdot \beta_k(i)(1 + K(i)) + \sum_{v_k \in e - S_1} \beta_k(i)} = 0$$

If  $v_j \in e \cap S_1$ ,

$$\begin{aligned} \lim_{i \rightarrow \infty} f_{e, v_j}(\alpha(i)) &= \lim_{i \rightarrow \infty} \frac{i \cdot \beta_j(i)(1 + K(i))}{\sum_{v_k \in e \cap S_1} i \cdot \beta_k(i)(1 + K(i)) + \sum_{v_k \in e - S_1} \beta_k(i)} \\ &= \lim_{i \rightarrow \infty} \frac{\beta_j(i)}{\sum_{k \in e \cap S_1} \beta_k(i)} = \bar{f}_{e \searrow S, v_j}(Y) \end{aligned}$$

$\alpha(i)$  converges to a point  $X \in \bar{Q}_S$ , and  $F(X) = Y$ .  $F$  is onto.  $\square$

**Theorem 46.**  $P_H$  forms a CW-complex with open cells  $\mathcal{E}_H$ .

*Proof.* By Propositions 40 and 41, Theorem 44, and Lemma 45, it suffices to show that  $P_{H \searrow S}$  has a dimension smaller than  $n - c$ . It suffices to show that  $H \searrow S$  has more components than  $H$ . Some edge in  $H$  has vertices in both  $S_1$  and  $V - S_1$ , since every component of  $H$  has at least one vertex in  $V - S_1$ , and  $S_1$  is non-empty. In  $H \searrow S$ , no vertex in  $S_1$  is in the same component as any vertex in  $V - S_1$ , as no edges have vertices in both. Any component in  $H$  which has a non-empty intersection with  $S_1$  is split into at least 2 components in  $H \searrow S$ .  $\square$

**Theorem 47.** *There is a bijection between the open cells  $C \in \mathcal{E}_H$  and the chains of non-empty subsets of the vertices,  $S, V = S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots \supseteq S_k = \emptyset$ , such that for  $0 \leq i < k$ , every vertex in  $S_i - S_{i+1}$  is connected to a vertex in  $S_{i-1} - S_i$  in the graph  $H - S_{i+1}$ .  $\bar{C}$  is isomorphic in the category of CW-complexes to  $P_{H \searrow S}$ .*

*Proof.* The statement is trivially true for the interior of  $P_H$ ;  $k = 0$ , and  $S$  is just  $S_0 = V$ . The proof is by induction on  $n - c$ . If  $n - c = 0$ , the interior is the only cell and  $V = S_0 \subsetneq S_1$  is the only chain, so the proof is complete. Suppose  $n - c > 0$ , and  $C$  is a cell other than the interior of  $P_H$ .  $C \subset \bar{Q}_{S(C)}$ , and by Lemma 45,  $\bar{Q}_{S(C)}$  is isomorphic in the category of CW-complexes to  $P_{H \searrow S(C)}$ , which has a smaller dimension. By induction,  $\bar{C}$  is isomorphic to  $P_{(H \searrow S(C)) \searrow \tilde{S}}$ , where  $\tilde{S}_1 \subsetneq S(C)$ . Let  $S$  be the chain  $V = S_0 \supseteq S(C) \supseteq \tilde{S}_1 \supseteq \dots \supseteq \tilde{S}_k$ .  $(H \searrow S(C)) \searrow \tilde{S} = H \searrow S$ ,

so  $\overline{C}$  is isomorphic in the category of CW-complexes to  $P_{H \setminus S}$ . For a given chain  $S$ , let  $C$  be the cell containing the element with the representative Cauchy sequence  $a(i) = \sum_{j=0}^k i^j \cdot \mathcal{V}_{S_j}$ .  $\square$

The following result allows a reduction of dimensionality in some special cases.

**Theorem 48.** *Let  $e \in E$  be an edge with  $|e| \geq 3$  vertices such that there are three edges  $e_1, e_2, e_3 \subset e$  each with  $|e| - 1$  vertices. Then  $P_H$  is isomorphic as a CW-complex  $P_{H-e}$ .*

*Proof.* It suffices to find a continuous bijective homeomorphism from  $\overline{f}(P_{H-e})$  to  $\overline{f}(P_H)$  which maps cells to cells. This homeomorphism, call it  $\mathbb{H}$ , can fix all of the coordinates which are shared by both  $\overline{f}(P_{H-e})$  and  $\overline{f}(P_H)$ . It suffices to prove that  $\mathbb{H}_{e,v_j}$  is continuous and bijective for every  $v_j \in e$  in order for  $\mathbb{H}$  to be a homeomorphism. The only inputs that will be used in  $\mathbb{H}_{e,v_j}$  will be the  $\overline{f}_{e_i,v_\ell}$  terms for  $i \in \{1, 2, 3\}$ . Without loss of generality, assume that  $H$  has  $|e|$  vertices, and  $e_1, e_2, e_3$ , and  $e$  are the only edges.

Without loss of generality, assume that  $e_i = e - v_i$  for  $i \in \{1, 2, 3\}$ . On the interior of  $\overline{f}(P_{H-e})$ ,  $\overline{f}$  is a bijection with  $\mathbb{R}_{>0}^n / \approx$ . The  $\approx$  equivalence relation is the same for both  $H$  and  $H - e$ , so  $\overline{f}_{e,v_j}$  can be applied to the quotient space of  $\mathbb{R}_{>0}^n$  over the equivalence relation  $\approx_{H-e}$ .  $\mathbb{H}_{e,v_j}$  applies this map to the interior of  $\overline{f}(P_{H-e})$ . Suppose  $X$  is on the boundary of  $\overline{f}(P_{H-e})$ . At least one of  $X$ 's coordinates is 0. First assume that  $X_{e_i,v_j} = 0$  for some  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ . Note that proof of this case is all that is required for the base case of  $|e| = 3$ , so we can induct on  $|e|$  after this case is proven. Let  $\mathbb{H}_{e,v_j}(X) = 0$ , and  $\mathbb{H}_{e,v_\ell}(X) = X_{e_j,v_\ell}$ . To see that this is continuous, let  $a(t)$  be a Cauchy sequence such that  $\lim_{t \rightarrow \infty} \tilde{f}(a(t)) = X$ .

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow \infty} \mathbb{H}_{e,v_j}(\tilde{f}(a(t))) = \lim_{t \rightarrow \infty} \tilde{f}_{e,v_j}(a(t)) \\ &= \lim_{t \rightarrow \infty} \frac{a_j(t)}{\sum_{k=1}^{|e|} a_k(t)} \leq \lim_{t \rightarrow \infty} \frac{a_j(t)}{\sum_{k \neq i} a_k(t)} = X_{e_i,v_j} = 0 \end{aligned}$$

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \mathbb{H}_{e, v_\ell}(\tilde{f}(a(t))) = \lim_{t \rightarrow \infty} \tilde{f}_{e, v_\ell}(a(t)) \\
& = \lim_{t \rightarrow \infty} \frac{a_\ell(t)}{\sum_{k=1}^{|e|} a_k(t)} = \lim_{t \rightarrow \infty} \frac{a_\ell(t)}{\sum_{k \neq j}^{|e|} a_k(t)} \frac{\sum_{k=1}^{|e|} a_k(t)}{\sum_{k=1}^{|e|} a_k(t)} = X_{e_j, v_\ell} (1 - X_{e_i, v_j}) = X_{e_j, v_\ell}
\end{aligned}$$

Now assume  $X_{e_i, v_j} \neq 0$  for all  $i, j \in \{1, 2, 3\}$ , but some other  $X$  coordinate is 0. In particular, there must be at least 4 vertices to require this case. Since  $X$  is on the boundary, there is an  $S_1$  so that the closure of  $C$ , the cell containing  $X$ , is isomorphic to  $P_{(H-e) \searrow S} = P_{H \searrow S - e \searrow S}$ . For any  $v_j \notin S_1$ ,  $X_{e_i, v_j} = 0$  for  $j \neq i$ . This implies that  $v_1, v_2, v_3 \in S_1$ .  $e_1 \cap S$ ,  $e_2 \cap S$ ,  $e_3 \cap S$  and  $e \cap S$  satisfy the conditions for induction with  $|e \cap S| < |e|$ .  $P_{H \searrow S - e \searrow S}$  is isomorphic as a CW-complex to  $P_{H \searrow S}$ , which is isomorphic to the closure of a cell  $D$  in  $P_H$ .  $\mathbb{H}(X)$  chains these isomorphisms together to map  $X$  to  $P_H$ .  $\square$

### 3.3 FROM GRAPHS TO ZONOTOPES

This section explores the connection between the CW-complex formed by simple graphs, and zonotopes with isomorphic CW-complexes. A zonotope is a special type of polytope which is the image of a hypercube under linear projection. An explicit map from  $P_H$  to a zonotope is given which maps  $k$ -cells to  $k$ -faces of the zonotope, thus preserving the CW-complex structure. This is done by taking the image of  $P_H$  under  $\bar{f}$ , which is clearly an isomorphism, and taking a linear projection onto a subspace of dimension  $n - c$ . In order for this to be an isomorphism in the category of CW-complexes, the linear projection needs to be one-to-one. It suffices to find a linear space of dimension  $m - n + c$  such that no two points in the image of  $P_H$  under  $\bar{f}$  have a difference which is in the linear space.

In a simple graph,  $m = 2|E|$ , as there are 2 coordinates in the image for each edge, one for each vertex in the edge. If  $e = (v_i, v_j)$ , then  $\bar{f}_{e, v_i}(X) = 1 - \bar{f}_{e, v_j}(X)$ .

This induces a linear projection down to  $\mathbb{R}^{|E|}$  by arbitrarily choosing one vertex for each edge. Essentially, this induces a direction on each edge, giving a directed graph  $\vec{H}$ , where the image of  $\vec{f}$  is restricted to the coordinates for the in-vertices of each directed edge. This restriction, call it  $\vec{f}$ , is an isomorphism. Another linear space of dimension  $|E| - n + c$  is needed to complete the linear map into a space of dimension  $n - c$ . The space used is a form of a cycle space on the directed graph  $\vec{H}$ .

For every cycle  $C$  in  $H$ , there are two directed cycles  $\vec{C}$ . Note that all of the edges are directed the same way around the cycle, and that this may conflict with the directions given to those edges in  $\vec{H}$ . Define the vector  $\mathbb{V}(\vec{C}) \in \mathbb{R}^{|E|}$  as the vector with  $\mathbb{V}_e(\vec{C}) = 0$  if  $e \notin C$ ,  $\mathbb{V}_e(\vec{C}) = 1$  if  $e$  is given the same direction in  $\vec{C}$  and  $\vec{H}$ , and  $\mathbb{V}_e(\vec{C}) = -1$  if it is given different directions. Let  $\mathcal{C}$  be the vector space generated by these  $\mathbb{V}(\vec{C})$ . Let  $T$  be a spanning forest of  $H$ . Every edge not in  $T$  induces a unique cycle in  $H$ , and choosing one directed cycle for each one generates vectors  $\mathbb{V}(\vec{C})$  for each of these cycles, which forms a linearly independent set.  $T$  has  $n - c$  edges, which implies that  $\mathcal{C}$  has dimension at least  $|E| - n + c$ . This will be proven to be exact when the dimension of the complementary space is discussed.

**Theorem 49.** *For any  $a, b \in P_H$ ,  $\vec{f}(a) - \vec{f}(b) \notin \mathcal{C}$*

*Proof.* For a cycle  $\vec{C}$ , say that the vector  $X$  matches  $\mathbb{V}(\vec{C})$  if for every  $e \in C$ ,  $X_e \neq 0$  and both  $X_e$  and  $\mathbb{V}_e(\vec{C})$  have the same sign. To see that  $\vec{f}(a) - \vec{f}(b)$  cannot match  $\mathbb{V}(C) \in \mathcal{C}$ , suppose for contradiction that it does. Let  $(c_0, c_1, \dots, c_k)$  be the vertices of  $C$ , with the edges directed from  $c_i$  to  $c_{i+1}$  in  $\vec{C}$ .  $a_i - b_i < a_{i+1} - b_{i+1}$  for all  $i$ . This implies  $a_0 - b_0 < a_k - b_k < a_0 - b_0$ , an obvious contradiction. It suffices to show that every vector in  $W \in \mathcal{C}$  matches some  $\mathbb{V}(\vec{C})$ .

Another way to look at this is to take  $W$  and make a directed graph  $G$  on the vertices of  $H$  where each edge with a positive coordinate in  $W$  takes the direction from  $\vec{H}$ , and each edge with a negative coordinate in  $W$  takes the opposite direction. Ignore any edge whose coordinate in  $W$  is 0. If  $G$  has a directed cycle, then  $W$

matches  $\mathbb{V}(\vec{C})$  for that cycle. It suffices to show that  $G$  always has a directed cycle, for which it suffices to show that  $G$  has no sinks. Pick a vertex  $v \in G$ . Assume  $v$  has at least one in-edge, otherwise it is not a sink. Given a basis  $B$  for  $\mathcal{C}$ , where each  $B_i = \mathbb{V}(\vec{C})$  for some directed cycle,  $W = \sum_i d_i B_i$ . Define the matrix  $M(v)$  for a vertex  $v$ , with a row for each edge incident to  $v$ , and a column for each  $B_i$ .  $M_{i,j}$  is 0 if the  $e_i \notin B_j$ , -1 if the edge is directed into  $v$  in  $B_j$ , and 1 if it is directed out. The edge  $e_i$  is directed into  $v$  in  $G$  iff  $\sum_j M_{i,j} d_j$  is negative, and directed out iff the sum is positive. Since the  $B_i$  are cycles, they have precisely 1 in edge and 1 out edge incident to  $v$ , so the columns of  $M$  sum to 0, and  $Md$  sums to 0. By assumption  $v$  has at least 1 edge directed in, so  $Md$  has at least 1 negative entry. It sums to 0, so it must also have a positive entry, and therefore an out edge.  $\square$

Projecting into the space complementary to  $\mathcal{C}$  is 1-1, and thus a bijection with the image. In order to define the function which projects into the complementary space, a basis of the complementary space can be found. Any basis can be used to construct an orthonormal basis using the Gram-Schmidt process. Projecting into the complementary space is then equivalent to taking inner products with this orthonormal basis.

The complementary space will be called  $\mathcal{B}$ . Recall the spanning forest  $T$ . For any edge in  $T$ , removing it breaks the component containing that edge into 2 components, call them  $A$  and  $B$ . Define the vector  $V_e \in \mathcal{B}$  as the vector with a 1 for edge  $e$  if it is directed from  $A$  to  $B$  in  $\vec{H}$ , -1 if it is from  $B$  to  $A$ , and 0 otherwise. As each of these cuts has a unique edge in  $T$ , these vectors are linearly independent, with a span of dimension  $n - c$ . Since they are normal to  $\mathcal{C}$ , they are contained in the complement.  $\mathcal{C}$  had dimension at least  $|E| - n + c$ , so  $\mathcal{B}$  must be the complement.

Projecting into  $\mathcal{B}$  is bijective, so  $P_H$  is isomorphic in the category of CW-complexes with its image. To see that the result of the projection is a zonotope, it is proven that  $P_H$  projects onto the image of the unit hypercube under the same projection.

**Theorem 50.**  $P_H$  and  $[0, 1]^m$  have the same projection in the cut space  $\mathcal{B}$ .

*Proof.* The proof is by induction on  $|E| - n + c$ . In the case that  $H$  is a forest there is no projection as there is no cycle space, and  $P_H$  maps onto the unit hypercube under  $\vec{f}$ .

Pick  $x$  in the image of the unit hypercube projected into  $\mathcal{B}$ , and let  $R$  be its pre-image; that is  $R$  is the intersection of the unit hypercube, and the affine space containing  $x + W$  for  $W \in \mathcal{C}$ . Let  $E'$  be the set of edges whose coordinates are either always 0 or always 1 in  $R$ . First, assume  $E' \neq \emptyset$ . Consider the graph  $H - E'$ . If  $A$  and  $B$  are separate components of  $H - E'$ , then either all edges from  $A$  to  $B$  in  $E'$  have coordinates of 1 in  $S$ , and edges from  $B$  to  $A$  have coordinates of 0, or the other way around. If this is not true, then there is a cycle  $C$  whose  $\mathbb{V}(\vec{C})$  can be added so that those edges have coordinates in  $(0, 1)$ . Note that for similar reasons, all edges in  $E'$  are between two components of  $H - E'$ . This induces a partial order on the components of  $H - E'$ , where  $A < B$  if there is a sequence of components  $A = A_1, A_2, \dots, A_k = B$  such that all of the edges in  $E'$  from  $A_i$  to  $A_{i+1}$  have coordinates that are always 1 in  $S$ . This is anti-symmetric, and transitivity is preserved by a similar condition on a cycle that would result. So the components of  $H - E'$  form a poset. For each component  $A$  of  $H - E'$ , define the height  $\eta(A)$  as one less than the length of the longest chain in  $H - E'$  with  $A$  as the maximal element. Extend this to  $\eta(v_i)$ , so that  $\eta(v_i) = \eta(A)$  if  $v_i \in A$ . Define the set  $S_i$  as the set of  $v_j$  such that  $\eta(v_j) \leq i$ . This defines a decreasing chain of sets of vertices  $S = \{V = S_0, S_1, \dots, S_k\}$ .  $H - E' = H \searrow S$ . By induction, there exists a point  $x \in P_{H \searrow S}$  which maps into the restriction of  $R$  to edges not in  $E'$ . By Theorem 47,  $S$  corresponds to a cell  $D \in \mathcal{E}_H$ , and the closure  $\bar{D}$  is isomorphic to  $P_{H \searrow S}$ . Let  $y \in \bar{D}$  be the image of  $x$  under this isomorphism.  $\vec{f}(y) \in R$ .

Now assume  $E' = \emptyset$ . There exists a point  $t \in R$  such that all of the edges are in  $(0, 1)$ . Pick a spanning forest  $T$  of  $H$ , and label the edges not in  $T$  as  $e_1, e_2, \dots, e_k$ .

For any  $t \in R$  there exists  $y \in P_H$  such that for all edges  $e \in T$ ,  $\vec{f}_e(y) = t_e$ . Call this point  $y(t)$ . For each edge  $e_i$  not in  $T$ , define  $g_i(t) = t_i - f_{e_i}(y(t))$ . It suffices to find a point in  $R$  where all of the  $g$ 's are 0.  $h_i(t)$  will be defined to map any point  $t \in R$  to a point in  $R$  where  $g_j(h_i(t)) = 0$  if  $j \leq i$ . The  $h_i$ 's are built up inductively, starting with  $h_0(t) = t$ .

Each  $e_i$  induces a cycle  $C_i$ , which induces a vector  $W_i \in \mathcal{C}$ ,  $W_i = \mathbb{V}(C_i)$ .  $h_i(t) = t + \sum_{j=1}^i \alpha_j V_j$  for some collection of  $\alpha$ 's. Let  $t$  be given. Let  $A_i(t)$  be the set of points in  $R$  of the form  $t + \sum_{j=1}^i \alpha_j V_j$ .  $A_{i-1}(t) \subseteq A_i(t)$ , and for any  $s \in A_i(t)$ ,  $A_i(s) = A_i(t)$ , and  $h_{i-1}(s), h_{i-1}(t) \in A_i(t)$ . Let  $B_i(t) = h_{i-1}(A_i(t))$ . The  $W_j$  are linearly independent, so there is a unique way to write each point in  $A_i(t)$  in the given form. Let  $\sim$  be an equivalence relation on  $A_i(t)$ , where two points are equivalent if they have the same  $\alpha_i$ .  $h_{i-1}$  is well-defined and continuous on  $A_i(t)/\sim$ .  $A_i(t)$  is the intersection of two closed sets, so it is closed. There exists a point with the minimum  $\alpha_i$  over all of  $A_i(t)$ , call it  $\beta$ . For any  $\epsilon > 0$ ,  $\beta - \epsilon W_i \notin R$ , so  $\beta_e = 0$  for some edge  $e \in C_i$ . Since  $h_{i-1}(\beta) \in A_i(t)$  has the same  $\alpha_i$  value, it too has a 0 for some edge in  $C_i$ .  $g_i(h_{i-1}(\beta)) \leq 0$ . Similarly, let  $\gamma \in A_i(t)$  be the point with the maximum  $\alpha_i$ , implying  $g_i(h_{i-1}(\gamma)) \geq 0$ . By the intermediate value theorem, there is a point in  $x \in A_i(t)/\sim$  such that  $g_i(h_{i-1}(x)) = 0$ .  $h_i(t) = h_{i-1}(x)$ .

Then  $h_k(t)$  is a point with all of the  $g_i$ 's 0, so it is the image of a point in  $P_H$ .  $h_k(t) \in R$ , and  $x$  is in the image of the projection of  $P_H$ . Therefore the projection is onto, as claimed.  $\square$

The resulting zonotope is the intersection of finitely many closed half-spaces. As the zonotope is full dimensional, the hyperplanes defining the half-spaces each intersect the zonotope in a space of dimension  $n - 1$ , where  $n$  is the dimension of the zonotope. The cells of the natural CW-complex on the zonotope results from labeling each point with the supporting hyperplanes it is incident to. The dimension of the cell is the dimension of the zonotope minus the number of supporting hyperplanes.

**Corollary 51.** *The CW-complex  $P_H$  is isomorphic as a CW-complex with the natural CW-complex of the zonotope formed under projection onto the cut space  $\mathcal{B}$ .*

*Proof.* It suffices to show that the image of an  $(n - c - k)$ -cell in the CW-complex is an  $(n - c - k)$ -face in the zonotope. Let  $F$  be an  $(n - c - k)$ -face of the zonotope.  $F$  is determined by  $k$  supporting hyperplanes. Let  $R$  be the pre-image of  $F$  in  $[0, 1]^{|E|-n+c}$ . The pre-image of the supporting hyperplanes of the zonotope must be supporting hyperplanes of the hypercube. A supporting hyperplane of the hypercube forces a given set of coordinates to be 1, and another set to be 0. As this hyperplane is the preimage of a hyperplane in  $\mathcal{B}$ , it is closed under the addition of any element of  $\mathcal{C}$ . As described in the previous proof, this means that the edges which are fixed at 0 and 1 induce a cut in  $H$ , as well as a poset on the resulting components. Let  $E'$  be the set of edges whose coordinates are either always 0 or always 1 in  $R$ . Because  $R$  has  $k$  supporting hyperplanes, and each must fix the coordinate of at least one edge which no other hyperplane does,  $H - E'$  has at least  $c + k$  components. Using the poset as in the previous proof, a chain  $S = \{V = S_0, \dots, S_\ell\}$  is defined. Let  $D \in \mathcal{E}_H$  be the cell corresponding to this chain. Every element of  $C$  is mapped into  $R$ , and any element mapped into  $R$  is in  $C$ . Thus there is a homeomorphism between  $C$  and  $F$ , so they are cells of the same dimension.  $\square$

### 3.4 SPECIAL CASES

As discussed above, the CW-complex  $P_H$  is isomorphic to the natural CW-complex formed by projection onto the cut space  $\mathcal{B}$ , complementary to the cycle space  $\mathcal{C}$ . A tree has no cycles, and thus no cycle space. As a result the zonotope for a tree on  $n$  vertices is the  $(n - 1)$ -hypercube.

Now consider the cycle graph  $C_n$ . The cycle space has only 1 cycle, so the cycle space is generated by the vector  $(1, \dots, 1)$ , assuming without loss of generality that the same directions were chosen for the cycle as a cycle and as the graph. The resulting

zonotope for  $P_{C_n}$  is the projection of the  $n$ -hypercube onto the complementary space.  $P_{C_3}$  is a hexagon, and  $P_{C_4}$  is a rhombic dodecahedron.

Finally, consider the complete graph  $K_n$ . The resulting zonotope is the  $(n - 1)$ -dimensional permutohedron. In order to prove this, it is simpler to project onto the canonical permutohedron in  $\mathbb{R}^n$ , which is defined as the convex hull of all permutations of  $(1, \dots, n)$ . An orthonormal basis is constructed to define the projection, which is then modified in order to map the hypercube into the hyperplane containing the permutohedron.

**Proposition 52.**  $P_{K_n}$  is isomorphic in the category of CW-complexes to the  $(n - 1)$ -dimensional permutohedron.

*Proof.* Define  $\vec{K}_n$  by labelling each vertex  $1, 2, \dots, n$ , and directing edges from larger numbers to smaller numbers. Define the tree  $T$  as the edges incident to  $v_n$ . For  $i > j$ , let  $\xi_{i,j}$  be the vector with a 1 in the coordinate for the edge from  $j$  to  $i$ , and 0 in all other coordinates. For  $1 \leq i \leq n$ , define

$$B_i = \sum_{j=1}^{i-1} \xi_{i,j} - \sum_{j=i+1}^n \xi_{j,i}.$$

If  $i \neq j$ ,  $B_i \cdot B_j = -1$ , and  $B_i \cdot B_i = n - 1$ . For  $1 \leq i \leq n - 1$ , the  $B_i$ 's form a basis of  $\mathcal{B}$ .

$$N_i = B_i + \frac{1}{i+1} \sum_{j=i+1}^{n-1} B_j$$

If  $k > i$ ,

$$N_i \cdot B_k = -1 + \frac{n-1}{i+1} - \frac{n-i-2}{i+1} = 0$$

The  $N_i$  are orthogonal.

$$\begin{aligned} N_i \cdot N_i &= N_i \cdot \left( B_i + \frac{1}{i+1} \sum_{j=i+1}^{n-1} B_j \right) = N_i \cdot B_i = n - 1 - \frac{n-i-1}{i+1} \\ &= \frac{ni - i + n - 1 - n + i + 1}{i+1} = \frac{ni}{i+1} \end{aligned}$$

$V_i = \sqrt{\frac{i+1}{ni}}N_i$  forms an orthonormal basis.  $U \in \mathbb{R}^n$  is the space where  $u \in U$  iff  $u \cdot (1, 1, \dots, 1) = 0$ . For  $1 \leq i \leq n$ , define  $D_i \in U$ .

$$D_i = (1, \dots, 1, -(n-1), 1, \dots, 1),$$

where the  $-(n-1)$  is in the  $i$ th position, and there are  $n-1$  ones. This is a linearly independent set for  $1 \leq i \leq n-1$ . For  $i \neq j$ ,

$$D_i \cdot D_j = -2\frac{n-1}{4} + \frac{n-2}{4} = \frac{-2n+2+n-2}{4} = -\frac{n}{4}$$

$$D_i \cdot D_i = \frac{n-1}{4} + \frac{(n-1)^2}{4} = \frac{n}{4}(n-1)$$

Just as with the  $B_i$ 's, an orthogonal set is formed by

$$M_i = D_i + \frac{1}{i+1} \sum_{j=i+1}^{n-1} D_j$$

$W_i = \sqrt{\frac{i+1}{ni}}M_i$  are not unit length, but are an orthogonal basis with all of the vectors the same length.

Let  $\alpha = (\frac{1}{2}, \dots, \frac{1}{2})$ . For any  $X$  in the unit hypercube there are unique  $a_i$  such that

$$X \in \alpha + \sum_{i=1}^{n-1} a_i V_i + \mathcal{C}.$$

$F$  is the function which maps  $X$  to the permutohedron. Let  $\beta = (\frac{n+1}{2}, \dots, \frac{n+1}{2})$ .

$$F(X) = \beta + \sum_{i=1}^{n-1} a_i W_i$$

Note that

$$F(\alpha + \sum_{i=1}^n b_i B_i + \mathcal{C}) = \beta + \sum_{i=1}^n b_i D_i$$

If  $\sum_{j=1}^n b_j = 0$ ,

$$B_i \cdot \sum_{j=1}^n b_j B_j = b_j(n-1) - \sum_{j \neq i} b_j = b_i n.$$

For any  $X$ ,  $\sum_{i=1}^n X \cdot B_i = 0$ , so  $X \in \sum_{i=1}^n \frac{X \cdot B_i}{n} B_i + \mathcal{C}$ .

$$\alpha \cdot B_i = \frac{1}{2} \left( \sum_{j=1}^{i-1} 1 - \sum_{j=i+1}^n 1 \right) = \frac{1}{2}(i-1-n-i) = i - \frac{n+1}{2}$$

$$\begin{aligned}
X &\in \alpha + \sum_{i=1}^n \left( \frac{n+1}{2n} - \frac{i}{n} + \frac{X \cdot B_i}{n} \right) B_i + \mathcal{C} \\
F(X) &= \beta + \sum_{i=1}^n \left( \frac{n+1}{2n} - \frac{i}{n} + \frac{X \cdot B_i}{n} \right) C_i \\
F_i(X) &= \frac{n+1}{2} - (n-1) \left( \frac{n+1}{2n} - \frac{i}{n} + \frac{X \cdot B_i}{n} \right) + \sum_{j \neq i} \left( \frac{n+1}{2n} - \frac{i}{n} + \frac{X \cdot B_i}{n} \right) \\
&= \frac{n+1}{2} - n \left( \frac{n+1}{2n} - \frac{i}{n} + \frac{X \cdot B_i}{n} \right) = i - X \cdot B_i
\end{aligned}$$

For  $F(X)$  to be in the permutohedron, it suffices to show that for every subset  $S$  of the numbers 1 to  $n$ ,

$$\sum_{i=1}^{|S|} i \leq \sum_{i \in S} F_i(X) \leq \sum_{i=1}^{|S|} n+1-i$$

Let  $S$  be given.

$$\begin{aligned}
\sum_{i \in S} F_i(X) &= \sum_{i \in S} (i - X \cdot B_i) = \sum_{i \in S} \left( i - \sum_{j < i} \xi_{i,j} \cdot X + \sum_{i < j} \xi_{j,i} \cdot X \right) \\
&= \sum_{i \in S} \left( i - \sum_{\substack{j < i \\ j \notin S}} \xi_{i,j} \cdot X + \sum_{\substack{i < j \\ j \notin S}} \xi_{j,i} \cdot X \right) \tag{3.1} \\
&\geq \sum_{i \in S} \left( i - \sum_{\substack{j < i \\ j \notin S}} 1 \right) = \sum_{i \in S} \left( 1 + \sum_{\substack{j < i \\ j \in S}} 1 \right) = \sum_{i=1}^{|S|} i
\end{aligned}$$

Starting again from Equation 3.1,

$$\begin{aligned}
&\leq \sum_{i \in S} \left( i + \sum_{\substack{i < j \\ j \notin S}} 1 \right) = \sum_{i \in S} \left( n - \sum_{i < j} 1 + \sum_{\substack{i < j \\ j \notin S}} 1 \right) = \sum_{i \in S} \left( n - \sum_{\substack{i < j \\ j \in S}} 1 \right) \\
&= \sum_{i=1}^{|S|} n+1-i
\end{aligned}$$

The image of the hypercube is contained in the permutohedron.

Let  $\sigma$  be a permutation of the numbers 1 to  $n$ . There is a point in  $P_{K_n}$  which corresponds to the Cauchy sequence  $A(k) = (k^{\sigma_1}, \dots, k^{\sigma_n})$ . Let  $X$  be the image of this sequence under  $\vec{f}$ .  $X \cdot \xi_{i,j} = 1$  for  $i > j$  if  $\sigma_i < \sigma_j$ , and  $X \cdot \xi_{i,j} = 0$  otherwise.

It is worth noting here that this corner of the hypercube is represented by the upper triangle of the permutation graph corresponding to  $\sigma$ .

$$\begin{aligned}
B_i \cdot X &= \sum_{j < i} \xi_{i,j} \cdot X - \sum_{i < j} \xi_{j,i} \cdot X \\
&= |\{j | j < i \text{ and } \sigma_i < \sigma_j\}| - |\{j | i < j \text{ and } \sigma_j < \sigma_i\}| \\
&= |\{j | j < i \text{ and } \sigma_i < \sigma_j\}| - |\{j | i < j \text{ and } \sigma_j < \sigma_i\}| \\
&\quad + |\{j | j < i \text{ and } \sigma_j < \sigma_i\}| - |\{j | j < i \text{ and } \sigma_j < \sigma_i\}| \\
&= |\{j | j < i\}| - |\{j | \sigma_j < \sigma_i\}| = i - 1 - (\sigma_i - 1) = i - \sigma_i \\
X &\in \alpha + \sum_{i=1}^n \left( \frac{n+1}{2n} - \frac{\sigma_i}{n} \right) B_i + \mathcal{C} \\
F(X) &= \beta + \sum_{i=1}^n \left( \frac{n+1}{2n} - \frac{\sigma_i}{n} \right) D_i \\
F_i(X) &= i - (i - \sigma_i) = \sigma_i.
\end{aligned}$$

The image of the hypercube contains the permutohedron. □

## CHAPTER 4

### APPLICATION TO RANDOM WALKS ON GRAPHS

#### 4.1 INTRODUCTION AND DEFINITIONS

This chapter will examine the use of graphical projective spaces as the ambient space for weights in the graph random walk problem examined in Chapter 2. In particular,  $G$  is a directed graph on  $n$  vertices with a starting vertex  $v_{\text{start}} = v_1$ , and an ending vertex  $v_{\text{end}} = v_n$ .  $H$  is the hypergraph on the vertices of  $G$ , where the edges of  $H$  are the out-neighborhoods of  $G$ . The vertices of  $G$  are given weights by identifying with a point  $W \in P_H$ . The probability of going from  $v_i$  to  $v_j$  is given by the function  $\bar{f}_{N^+(v_j), v_i}(W)$ .

#### 4.2 OCCUPATION TIMES

Let  $A$  be the adjacency matrix of  $G$ , that is  $A_{i,j} = 1$  if  $v_i \rightarrow v_j$ .  $\tilde{A}$  is the adjacency matrix for  $G$  with an added edge from  $v_n$  to  $v_1$ . Define the probability matrix  $\tilde{P}$  as  $\tilde{P}_{i,j} = \bar{f}_{N^+(v_j), v_i}(W)$ . The expected number of visits is given by the unique eigenvector  $D$  with  $D_n = 1$  such that  $\tilde{P}D = D$ . Alternatively, let  $\bar{P}$  be the  $(n-1) \times (n-1)$  leading principal submatrix of  $\tilde{P}$ , and let  $\bar{D}$  be the first  $n-1$  entries of  $D$ .  $\bar{D} = (I - \bar{P})^{-1}\chi_1$ . Thus as long as  $I - \bar{P}$  is invertible,  $D$  is defined for  $W$ , and by continuity is the limit of the sequence  $D(i)$  formed by any sequence  $W(i)$  converging to  $W$ . The invertibility of  $I - \bar{P}$  is linked with the irreducibility of  $\tilde{P}$ . By the Perron-Frobenius theorem, since  $\tilde{P}$  is non-negative and all columns sum to 1, it has 1 as an eigenvalue with multiplicity 1 iff it is irreducible.

The irreducibility of  $\tilde{P}$  when  $W \in \mathbb{R}_{>0}^n$  comes from the fact that every vertex is on a path from  $v_1$  to  $v_n$ , along with the added edge from  $v_n$  to  $v_1$ . On the boundary of  $P_H$ , every face produces a chain induced hypergraph of  $H$ ,  $H \searrow S$ .  $H \searrow S$  is the neighborhood hypergraph of  $G'$ , where  $G'$  is obtained from  $G$  by removing all directed edges which have probability 0 at  $W$ . Clearly  $\tilde{P}(W)$  will be irreducible iff  $G'$  also has the property that every vertex is on a path from  $v_1$  to  $v_2$ . In this case  $\tilde{P}$  is clearly defined, so we can get  $I - \bar{P}$ , and it will be invertible by the Perron-Frobenius Theorem. The domain for the process will be the faces of  $P_H$  which give such  $G'$ . Note that the vector of expected occupation times  $D$  will still be in  $\mathbb{R}_{>0}^n$ , as each vertex is still being visited.

### 4.3 FINDING THE WEIGHTS

Some definitions are recalled to state the results. The trace of a walk  $\omega$  at  $v_i$  is defined as  $\text{tr}_i(\omega)$ , the number of times  $\omega$  visits  $v_i$ . Then  $\text{tr}(\omega)$  is the vector giving the number of visits to each vertex. The set of proper walks  $\Lambda$  is the set of walks which start at  $v_1$ , end at  $v_n$ , and only visit  $v_n$  once. For  $\omega \in \Lambda$ ,  $p(\omega)$  is the probability of the walk  $\omega$ . A set  $\Omega \subseteq \Lambda$  covers an edge  $e \in G$  if there exists a walk  $\omega \in \Omega$  such that  $\omega$  uses the edge  $e$ .  $\Omega$  covers all edges of  $G$  if such an  $\omega$  exists for each edge in  $G$ .  $D$  is a convex combination of traces of walks, so it is contained in the convex hull of the traces of proper walks, this convex hull is called  $\mathcal{C}$ . The minimal hyperplane in  $\mathbb{R}^n$  which contains  $\mathcal{C}$  is the affine hull of the traces of  $\Lambda$ , called  $\mathcal{H}$ . The hypergraph  $H$  is the hypergraph on the same vertices as  $G$  such that  $e \subseteq V(G)$  is an edge in  $H$  iff there exists  $v_k$  such that  $e = N^+(v_k)$ .  $c = c(G)$  is the number of components of  $H$ , and  $C_i$  is the set of vertices in the  $i$ th component of  $H$  for  $1 \leq i \leq c$ .  $R \subseteq V$  is a set of representatives of the components, that is  $|R \cap C_i| = 1$ .  $R(v_i)$  is used to mean the representative of the component containing  $v_i$ .  $\xi_S = 1$  if  $v_1 \in S$ , and 0 otherwise for

$S \subseteq V$ , with  $\xi_i = \xi_{\{v_i\}}$ .

$$F_i(W) = D_i - \xi_i - \sum_{j \in N^-(v_i)} D_j \bar{f}_{N^+(v_j), v_i}(W)$$

The following theorem was proven in Chapter 2 for the case when the weights are in  $\mathbb{R}_{>0}^n$ .

**Theorem 3.** *The following are equivalent*

1. *There exists a set of weights  $W$  such that  $D$  is the vector of expected occupation times on  $G$ .*
2.  *$D$  is in the relative interior of  $\mathcal{C}$  with respect to  $\mathcal{H}$ .*
3. *There exists  $\Omega \subseteq \Lambda$  which covers all edges of  $G$  such that  $D$  is a strict convex combination of the traces of walks in  $\Omega$ .*
4. *For every  $1 \leq i \leq c$  and for every  $S \subseteq C_i$*

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S$$

*with equality iff  $S = C_i$  or  $S = \emptyset$ .*

5. *For every  $S \subseteq V$ ,*

$$\sum_{v_j \in S} D_j \leq \sum_{v_j \in N^-(S)} D_j + \xi_S$$

*with equality iff  $S = \cup_{i \in \sigma} C_i$  for some  $\sigma \subseteq [c]$*

6. *For every set of representatives,  $R$ , there exists a set of weights  $W$  such that  $F_i(W) \leq 0$  for all  $i \in V - R$ .*

Very little about these properties change when the weights are allowed to be chosen from the sections of  $P_H$  for which the expected occupation time is defined.

The following theorem gives a list of equivalent properties for this case.

**Theorem 4.** *The following are equivalent*

1. *There exists a set of weights  $W \in P_H$  such that  $I - \bar{P}(W)$  is invertible, and  $D \in \mathbb{R}_{>0}^n$  is the vector of expected occupation times on  $G$ .*
2.  *$D$  is in the intersection of  $\mathbb{R}_{>0}^n$  and  $\mathcal{C}$*
3. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path such that there exists  $\Omega \subseteq \Lambda$  which covers all edges of  $G'$  such that  $D$  is a strict convex combination of the traces of walks in  $\Omega$ .*
4. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path, with corresponding graph  $H'$  with  $c'$  components  $C'_i$  such that for  $1 \leq i \leq c'$  and for every  $S \subseteq C'_i$*

$$\sum_{v_j \in S} D_j - \xi_S \leq \sum_{v_j \in N^-(S)} D_j,$$

*with equality iff  $S = C'_i$  or  $S = \emptyset$ .*

5. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path, with corresponding graph  $H'$  with  $c'$  components  $C'_i$  such that for every  $S \subseteq V$ ,*

$$\sum_{v_j \in S} D_j - \xi_S \leq \sum_{v_j \in N^-(S)} D_j,$$

*with equality iff  $S = \cup_{i \in \sigma} C'_i$  for some  $\sigma \subseteq [c]$*

6. *There exists a subgraph  $G'$  of  $G$  where every vertex is on a  $v_1$  to  $v_n$  path, with corresponding hypergraph  $H'$  such that for every set of representatives of  $H'$ ,  $R$ , there exists a set of weights  $W$  such that  $F_i(W) < 0$  for all  $i \in V - R$ .*
7. *There exists a set of weights  $W \in \mathbb{R}_{>0}^n$  such that  $D$  is the expected occupation times on  $G'$ .*

*Proof.* 1  $\Rightarrow$  2: By assumption  $D \in \mathbb{R}_{>0}^n$ . It suffices to show that  $D$  is in the convex hull of the traces of the proper walks on  $G$ .  $D$  is vector of expected occupation times,

so it is the sum of the traces of the walks times the probabilities of the walks. This implies that it is in the convex hull of these traces.

2  $\Rightarrow$  3: By assumption  $D$  is the convex combination of traces of some set of proper walks. Let  $G'$  be the subgraph of  $G$  containing only edges which are used in these walks.

3  $\Rightarrow$  4  $\Rightarrow$  5  $\Rightarrow$  6  $\Rightarrow$  7: These all follow from applying Theorem 3 to  $G'$ .

7  $\Rightarrow$  1: The out-neighborhood graph of  $G'$  is  $H' = H \setminus S$  for some  $S$ . The weights  $W \in P_{H \setminus S}$ , but  $P_{H \setminus S}$  is isomorphic to a cell of  $P_H$ . This isomorphism maps  $W$  into  $P_H$ , but the image produces the same probability matrix, and therefore returns the same vector of expected occupation times.  $\square$

Given a vector  $D$  of expected occupation times, a subgraph  $G'$  of  $G$  can be constructed. Applying our original algorithm to  $D$  on  $G'$  gives weights in  $P_{H \setminus S}$ , from which can be mapped into  $P_H$ . So the only remaining question is how to identify  $G'$  from  $D$ .

$G'$  can be identified from the inequalities from properties 4 and 5. Applying the inequalities to the components of  $H$ , weak inequality still holds, but equality might hold with a set other than a component, call this set  $S$ .  $S$  will be a component of  $H'$ . For any edge containing vertices in  $S$  and  $C_i - S$ , replace them with edges containing only the vertices in  $S$ .  $S$  becomes its own component, and  $C_i$  is broken up into different components. Repeating this process for any  $S$  which is not a component of  $H$  ends with  $H'$ . This process of restricting the edges in  $H$  corresponds to removing any edges in  $G$  from  $N^-(S)$  to  $C_i - S$ .

## CHAPTER 5

### OPEN PROBLEMS

There are several remaining questions related to those explored here.

1. The biggest missing piece is still the uniqueness of the weights which give a particular vector of expected occupation times; uniqueness is desired over  $\mathbb{R}_{>0}^n / \approx$  as any two weights which are  $\approx$  have the same probability matrices.
2. If the weights are unique in  $\mathbb{R}_{>0}^n / \approx$ , does this also hold for  $P_H$ ?
3. There are many ways to extend the domain of the process of finding expected occupation times to the entirety of  $P_H$ . Essentially this is done by allowing the number of visits to be 0 or  $\infty$ . In particular, this can be done by inducing some graph  $J$  on the vertices, and allowing the expected occupation times to come from  $P_J$ . Is there a way to do this which allows a set of weights in  $P_H$  to be recovered?
4.  $\mathbb{R}_{>0}^n$  is a semi-group under addition, and is metric space. Both of these properties are retained in  $P_H$  for any  $H$ . What other nice properties does  $P_H$  have?
5. When  $H$  is the hypergraph on  $n$  vertices with the one  $n$ -vertex edge,  $P_H$  is the projectivization of the closed first orthant. What properties of projective spaces do the other  $P_H$ 's have?
6. The graphical projective spaces induce a correspondence between simple graphs and zonotopes. There are many interesting questions that arise out of this. How can the number of faces of a particular dimension be calculated? How are the

zonotopes connected with other graph metrics? The 1-skeleton of the zonotopes is itself a simple graph. What connections are there between the graph  $G$  and the 1-skeleton graph of the zonotope  $P_G$ ?

7. It seems like there is a similar connection between the hypergraphs in general and zonotopes. Can the correspondence noted between simple graphs and zonotopes be extended to hypergraphs?
8. The CW-structure seems to be visible in the image of  $P_H$ , where the cells are smooth, and have singularities separating them. Can this be formalized when  $H$  is a graph? When it is a hypergraph?
9. The zonotope images of  $P_H$  when  $H$  is a graph have edges of equal length. The images when  $H$  is a hypergraph do not seem to; they seem to present variations, even when they are the same as CW-complexes. Is there an alternate formulation, or additional structure that can be taken into account to describe the spaces?

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