Solving Simple Japanese Ladder Games

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In this article we introduce a simpler version of Japanese ladder game. The mathematics of this game is discussed and an algebraic method is also introduced to solve this game.

Japanese Ladder

If you bought five Christmas gifts for your five children, how would you let them pick the gift without fighting with one another? A mathematical technique suggested by Lange and Miller in [5] might be a good solution to save you from a headache. On a paper draw five vertical lines. Randomly write the children's names at the top of each line, and the five gifts at the bottom of the lines. Cover the names and the gifts, and let each child randomly draw some horizontal rungs connecting adjacent vertical lines. Once done, this structure determines which child will receive which gift. To decode this matching, each child starts from the top of the vertical line that has his or her name on it. He or she then traces the vertical line downward until a horizontal line is met. He or she then follows the horizontal line to another vertical line, and then keep going downward. Repeat this process, until the end of a vertical line is reached. The gift at the bottom of that line therefore belongs to this child.

This ancient technique is very popular in Asia, and is usually used to represent a random permutation. Chinese call it "Ghost Leg" (畫鬼腳), Korean "Ladder Climbing" (사다리타기), and Japanese "Budda's Lots" (Amidakuji). The name "Japanese Ladder" was first raised in an earlier paper [2] by Dougherty and Vasquez, and is then adopted in our paper.

We first start with some terminologies. A Japanese ladder consists of several vertical lines and several horizontal bars, or rungs, connecting two adjacent vertical lines. From the top of each vertical line a path is traced through the ladder using the following three rules:

1. *When tracing a vertical line, continue downwards until an end of a rung is reached, then continue along the rung.*
2. *When tracing a rung, continue along it until the end of the rung is reached, then continue down the vertical line.*
3. *Repeat steps 1 and 2 until the bottom of a vertical line is reached.*

The example shown in figure 1 (a) is a Japanese ladder with three vertical lines and three rungs. The paths of tracing these three vertical lines are provided in figure 1 (b), (c), and (d).

For any Japanese ladder, a sequence of objects is placed at the top of these vertical lines, and a random rearrangement of these objects at the bottom. If all the necessary rungs are placed at the right place, this ladder structure creates an "one-to-one" and "onto" mapping from the top sequence to the bottom rearrangement (see [2] and [5]). To make it easier to explain in this paper, we will use consecutive numbers to indicate the top sequence, and we will just call it a sequence. We will also use these consecutive numbers to indicate the order of the vertical lines if it does not cause any confusion. The rearrangement of these numbers at the bottom of the ladder will then be simply called a permutation.

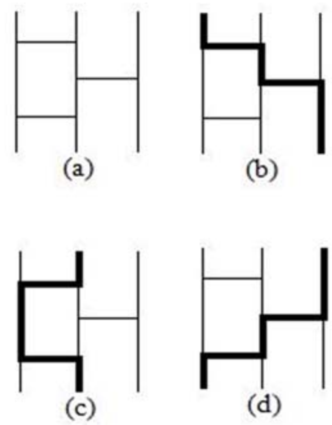

Figure 1.

A *k*-cycle is a permutation of k elements. Let a_i be a number in the top sequence for every *i*. Then the *k*-cycle $(a_1, a_2, \cdots, a_k)$ permutes the involving k numbers as follow: $a_1 \to a_2$, $a_2 \to a_3$, $\cdots$, $a_{k-1} \to a_k$, and $a_k \to a_1$. The composition of cycles will be referred as a product, and the composition symbol will be omitted. For example, $(1,2) \circ (2,3) = (1,2)(2,3)$. Since it is a composition after all, we follow the same convention we used in any compositions, reading it from right to left. Therefore in the example (1,2)(2,3), the net result is, 2 → 3, 3 → 1 (3 → 2 in the first (right) cycle, then 2 → 1 in the second (left) cycle), and 1 → 2. Two cycles are said to be disjoint if and only if they do not have any common elements. It is also known that the product of disjoint cycles is commutative. For more properties of *k*-cycle, we refer to Durbin [3], section 6.

One way, and the most common way, to play a Japanese ladder game is to place a random permutation at the bottom of a Japanese ladder without any rungs, and the player needs to create a minimum number of rungs to match the top sequence to the bottom permutation. A Japanese ladder with a minimum number of rungs that create a correct match will be called a minimum solution. The mathematics of this game has been discussed by Dougherty and Vasquez in [2] and by Lange and Miller in [5]. It is given in [5] that the minimum solution of a Japanese ladder game, in general, is not unique.

In the next section, we introduce a modified version of this game, called Simple Japanese Ladder Game. The mathematics of a simple Japanese ladder game is similar to, yet different from, the one of a normal version, and it will be discussed in section 2 as well. An algebraic method to find the minimum solution of this game will also be introduced.

Simple Japanese Ladder Game

To play a simple Japanese ladder game, the player also needs to create a Japanese ladder with minimum number of rungs to match the top sequence and the bottom permutation. Only this time, the rungs are allowed to cross over vertical lines. That means a rung may connect any two vertical lines, not just adjacent ones. The three rules for tracing a simple Japanese ladder remain the same. Just remember, when tracing a vertical line we turn only when we meet the end of a rung, not the middle of a rung. In figure 2 an example of simple Japanese ladder with two cross-over rungs is provided in (a). The paths of tracing each vertical line are shown in (b), (c), (d), and (e). Readers may easily see that, with the new condition of rungs inserted, a simple Japanese ladder game requires fewer rungs in its minimum solution. In figure 3, with the same permutation at the bottom, we show a solution of a Japanese ladder game in (a), and a solution of a simple Japanese ladder game in (b). That's why we call the modified version a "simple" one. Like a normal version Japanese ladder game, the minimum solution of a simple Japanese ladder game, in general, is not unique either.

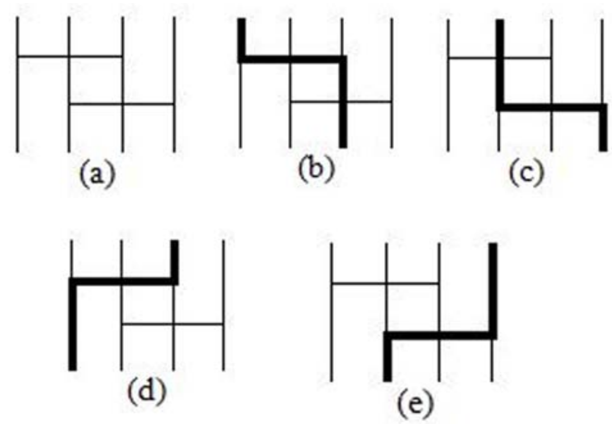

Figure 2.

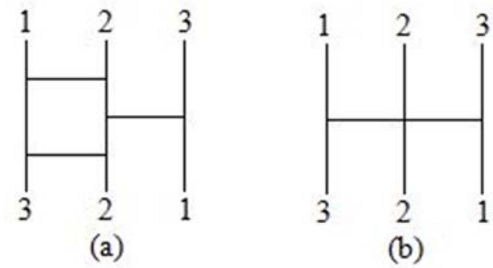

Figure 3.

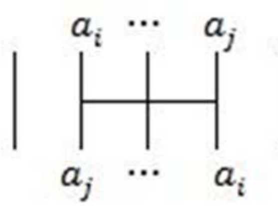

Figure 4.

Mathematics of Simple Japanese Ladder

The first fact we notice about a simple Japanese ladder is, a simple Japanese ladder also creates a one-to-one and onto mapping. From the top of any vertical line a number can be traced to the spot at the bottom that labeled the same number. But if we trace the bottom number backwards, meaning change the downwards rule to the upwards rule, we will go back to the same spot that we start with. That means this mapping is invertible. And we already know that any invertible mapping is one-to-one and onto. Another discussion about this fact, appropriate for both the Japanese ladder and the simple Japanese ladder, can be found in Lange-Willer [5], section 3.

We next will focus on the mathematics of rungs. In figure 4 we can see that when we trace two objects from the top to the bottom, they switch places when hitting the ends of a rung. So what a rung does is to create a transposition of two objects. We will use a 2-cycle to indicate this transposition. In the above case, it will be noted as (a_i, a_j), meaning the number at the a_ith vertical line is switched to the position at the a_jth line (and the a_jth number is switched to the a_ith place.) It is trivial then, that the mapping of a Japanese ladder (or a simple Japanese ladder) with n rungs is actually a combination of n transpositions, hence a product of n 2-cycles.

Theorem 1. *Any permutation in a simple Japanese ladder game can be written as a single cycle, or a product of disjoint cycles. Despite the order of the cycles, the product is unique.*

Proof. We may safely assume that any permutation in a simple Japanese ladder game is a finite permutation. In that case the first part of the theorem is apparently true according to [3] p.36. We will demonstrate one example here showing readers how to find the cycles. Let {5,3,6,1,4,2} be the permutation of the sequence {1,2,3,4,5,6}. We start our (first) cycle with the smallest number, 1. It is moved from the first vertical line to the fourth vertical line, so the cycle starts with 1 → 4. Now we examine the number 4. It is moved from the fourth line to the fifth line, so the cycle continues with 4 → 5. Our next target, number 5, is moved from the fifth line to the first line, so we have 5 → 1. The last mapping moves a number back to 1, the number we started with, hence closes the cycle (1,4,5). If the cycle includes every number in the permutation, it is the only cycle in the permutation. If there are other numbers not in the cycle, we then repeat the process to find other cycles. In our example, we still have numbers 2, 3, and 6 left, so we start the second cycle with the smallest remaining number, the number 2. We notice that 2 maps to 6, 6 maps to 3, and then 3 maps back to 2. So the second cycle is (2,6,3). We now see that the permutation {5,3,6,1,4,2} is the result of a product of two disjoint cycles, (1,4,5)(2,6,3). By the way, if a number is not moved (a *1*-cycle, the number maps to itself), it can be omitted in the product. For example, the permutation {6,5,3,1,4,2} can be written as the product $(1,4,5,2,6)(3)$, which is equal to the single cycle $(1,4,5,2,6)$.

Assume that a permutation can be written as two different products of disjoint cycles, noted $C_1C_2 \cdots C_k$ and $D_1D_2 \cdots D_l$, where C_i's and D_j's are cycles. Removing all the same cycles from these two products, we have the remaining two products still equal. $C'_1C'_2 \cdots C'_m = D'_1D'_2 \cdots D'_n$. Consider C'_i and D'_j from these two new products that C'_i and D'_j are not disjoint. Since $C'_i \neq D'_j$, there must be a common number in these two cycles that maps to two different numbers. This also reflects in their products. In the products $C'_1C'_2 \cdots C'_m$ and $D'_1D'_2 \cdots D'_n$, there is at least one number mapping to two different numbers accordingly. It contradicts that $C'_1C'_2 \cdots C'_m = D'_1D'_2 \cdots D'_n$. This implies, our assumption that a permutation can be written as two different products of disjoint cycles is not true. That proves the uniqueness. ∎

Theorem 2. *Any n-cycle can be written as a product of (n-1) 2-cycles. Moreover, (n-1) is the least number of 2-cycles needed.*

Proof. The first part of the theorem has been mentioned in many Algebra books, including [3]. It is quite self-evident, for we have known that for any *n*-cycle $(a_1, a_2, \cdots, a_n)$, it can be written as $(a_1, a_2)(a_2, a_3) \cdots (a_{n-1}, a_n)$, which is a product of (n-1) 2-cycles.

Assume that for an *n*-cycle, it can be written as a product of k 2-cycles where $k < (n-1)$. Since the product of these k 2-cycles can be merged into one *n*-cycle, each of the 2-cycle cannot be completely disjoint to all other 2-cycles. It must contains at least one element that is in common with another 2-cycle, and these two can therefore be merged into a 3-cycle. For the same reason this 3-cycle should be able to merge into a 4-cycle with another 2-cycle. Repeating this process we will then end up with a (k+1)-cycle. Since $(k+1) < n$, it contradicts our original assumption that it is a re-statement of an *n*-cycle. ∎

We want to point out that in the proof the mentioned product of (n-1) 2-cycles is not unique. For example, the same *n*-cycle $(a_1, a_2, \cdots, a_n)$ can also be written as $(a_1, a_n)(a_1, a_{n-1}) \cdots (a_1, a_2)$.

With these two results, we then can develop a strategy to solve a simple Japanese ladder game.

Solving Simple Japanese Ladder Game

To make it easier to understand, in this section we will use examples to explain each steps. For any simple Japanese ladder game, we first write the bottom permutation as a cycle or a product of disjoint cycles. In figure 5, we use the example used in Theorem 1, {5,3,6,1,4,2}, to indicate the bottom permutation of the top sequence {1,2,3,4,5,6}. We already know that we can write the permutation as a product of two disjoint 3-cycles $(1,4,5)(2,6,3)$.

Second, we will write the single *k*-cycle, or any individual *k*-cycle in the product, to a product of (k-1) 2-cycles. In the example mentioned above, $(1,4,5) = (1,4)(4,5)$ and $(2,6,3) = (2,6)(6,3)$. Therefore, $(1,4,5)(2,6,3) = (1,4)(4,5)(2,6)(6,3)$.

Third, for each 2-cycle, draw the corresponding rung in the order of the cycles from the right to the left. The four rungs corresponding to the four 2-cycle product $(1,4,5)(2,6,3) = (1,4)(4,5)(2,6)(6,3)$ is shown in figure 5.

Notice that Theorem 1 guarantees us that we can definitely find a solution of any simple Japanese ladder game. And Theorem 2 guarantees us that the solution we found using this method is a minimum solution.

We already mentioned that an *n*-cycle can be written as a product of 2-cycles in different ways. And different products will create different simple Japanese ladders. In figure 6 we provide another minimum solution of the same simple Japanese ladder game according to a different product of 2-cycles: $(1,4,5)(2,6,3) = (1,5)(1,4)(2,3)(2,6)$.

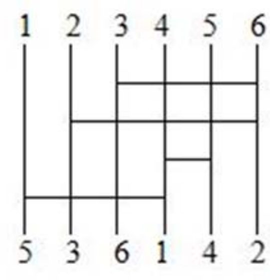

Figure 5.

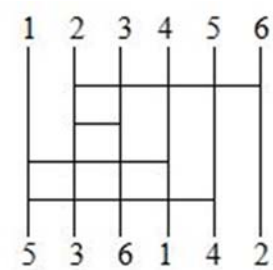

Figure 6.

Remarks

The method discussed in this article is just one of many ways to solve a simple Japanese ladder game. We have found that people playing the game use a variety of strategies. Some like to start from the first number that is moved from the left, and then continue to the next moved number until all numbers have been handled. Some players first identify the number that moves the farthest and fix that first. The second and the third farthest numbers then be taken care of in order, until all the moved numbers are fixed. If handled well, the player may also find a minimum solution using these techniques.

Notes and references

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