An improved Fibonacci inequality

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Fibonacci numbers and Fibonacci sequences play a key role in many areas of mathematics and other sciences. Many inequalities satisfied by Fibonacci sequences have been established. In this paper we prove a new Fibonacci inequality using Candido's identity.

Introduction

In this paper, we consider a problem in [1]. The problem posed is to prove that for every natural number n,

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)^2 > 81$$

where F_n is the n^{th} Fibonacci as defined in the section below.

We prove much stronger inequality:

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)^2 > 100.$$

Existing identity and inequality

Definition 1. The n^{th} Fibonacci number is defined by

$$F_n = F_{n-1} + F_{n-2}$$

where $F_0 = 0,\ F_1 = F_2 = 1\ and\ n \geq 2\ is\ a\ natural\ number.$

Lemma 2. (Candido's identity ([2], [3], [4]))
For every natural number n,

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 = 2\,(F_n^4 + F_{n+1}^4 + F_{n+2}^4).$$

Lemma 3. For every natural number n,

$$\frac{F_n}{F_{n+1}} \geq \frac{1}{2}.$$

Auxillary equations

In this section we give a proof of some auxillary equations that we use to obtain our main result.

Lemma 4. For every natural number n,

$$\left(\frac{F_{n+1}}{F_n}\right)^2 = 1 + 2\left(\frac{F_{n-1}}{F_n}\right) + \left(\frac{F_{n-1}}{F_n}\right)^2.$$

Proof.

$$\left(\frac{F_{n+1}}{F_n}\right)^2 = \left(\frac{F_n + F_{n-1}}{F_n}\right)^2$$

$$= \left(1 + \frac{F_{n-1}}{F_n}\right)^2$$

$$\left(\frac{F_{n+1}}{F_n}\right)^2 = 1 + 2\left(\frac{F_{n-1}}{F_n}\right) + \left(\frac{F_{n-1}}{F_n}\right)^2. \blacksquare$$

Lemma 5. For every natural number n,

$$\left(\frac{F_{n+2}}{F_n}\right)^2 = 4 + 4\left(\frac{F_{n-1}}{F_n}\right) + \left(\frac{F_{n-1}}{F_n}\right)^2.$$

Proof.

$$\left(\frac{F_{n+2}}{F_n}\right)^2 = \left(\frac{2F_n + F_{n-1}}{F_n}\right)^2$$

$$= \left(2 + \frac{F_{n-1}}{F_n}\right)^2$$

$$= 4 + 4\left(\frac{F_{n-1}}{F_n}\right) + \left(\frac{F_{n-1}}{F_n}\right)^2. \blacksquare$$

Result

In this section we give a proof of the new inequality which is our main result.

Theorem 6. (Main inequality)
For every natural number n,

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)^2 > 100.$$

Proof.

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)^2$$

$$= (F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)^2$$

(by Lemma 2.)

$$= \left(3 + \frac{F_{n+1}^2 + F_{n+2}^2}{F_n^2} + \frac{F_n^2 + F_{n+2}^2}{F_{n+1}^2} + \frac{F_n^2 + F_{n+1}^2}{F_{n+2}^2}\right)^2$$

$$= \left(3 + \left(\frac{F_{n+1}}{F_n}\right)^2 + \left(\frac{F_{n+2}}{F_n}\right)^2 + \left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_{n+2}}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2 + \left(\frac{F_{n+1}}{F_{n+2}}\right)^2\right)^2$$

$$= \left(9 + \frac{6F_{n-1}}{F_n} + \frac{2F_n}{F_{n+1}} + 2\left(\frac{F_{n-1}}{F_n}\right)^2 + 2\left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2 + \left(\frac{F_{n+1}}{F_{n+2}}\right)^2\right)^2$$

(by lemma 4 – 5.)

$$> \left(10 + \frac{6F_{n-1}}{F_n} + 2\left(\frac{F_{n-1}}{F_n}\right)^2 + 2\left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2 + \left(\frac{F_{n+1}}{F_{n+2}}\right)^2\right)^2$$

(by lemma 3.)

$$> 100. \blacksquare$$

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