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An improved Fibonacci inequality

Cameron T. Cooper
Division of Mathematics and Science, University of South Carolina Salkehatchie, Allendale, SC

Fidele F. Ngwane
Division of Mathematics and Science, University of South Carolina Salkehatchie, Allendale, SC

Wei-Kai Lai
Division of Mathematics and Science, University of South Carolina Salkehatchie, Allendale, SC

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Abstract
Fibonacci numbers and Fibonacci sequences play a key role in many areas of mathematics and other sciences. Many inequalities satisfied by Fibonacci sequences have been established. In this paper we prove a new Fibonacci inequality using Candido's identity.

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Cameron T. Cooper, Dr. Fidele F. Ngwane and Dr. Wei-Kai Lai

Division of Mathematics and Science, University of South Carolina Salkehatchie, Allendale SC 29810

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Introduction

In this paper, we consider a problem in [1]. The problem posed is to prove that for every natural number n,

$$2(F_n^4+F_{n+1}^4+F_{n+2}^4)\left(\frac{1}{F_n^2}+\frac{1}{F_{n+1}^2}+\frac{1}{F_{n+2}^2}\right)^2>81$$

where F_n is the n^{th} Fibonacci as defined in the section below.

We prove much stronger inequality:

$$2(F_n^4+F_{n+1}^4+F_{n+2}^4)\left(\frac{1}{F_n^2}+\frac{1}{F_{n+1}^2}+\frac{1}{F_{n+2}^2}\right)^2>100.$$

Existing identity and inequality

Definition 1. The n^{th} Fibonacci number is defined by

$$F_n=F_{n-1}+F_{n-2}$$

where $F_0=0,\ F_1=F_2=1\ and\ n\geq 2\ is\ a\ natural\ number.$

Lemma 2. (Candido's identity ([2], [3], [4]))
For every natural number n,

$$(F_n^2+F_{n+1}^2+F_{n+2}^2)^2=2\,(F_n^4+F_{n+1}^4+F_{n+2}^4).$$

Lemma 3. For every natural number n,

$$\frac{F_n}{F_{n+1}}\geq\frac{1}{2}.$$

Auxillary equations

In this section we give a proof of some auxillary equations that we use to obtain our main result.

Lemma 4. For every natural number n,

$$\left(\frac{F_{n+1}}{F_n}\right)^2=1+2\left(\frac{F_{n-1}}{F_n}\right)+\left(\frac{F_{n-1}}{F_n}\right)^2.$$

Proof.

$$\left(\frac{F_{n+1}}{F_n}\right)^2=\left(\frac{F_n+F_{n-1}}{F_n}\right)^2$$

$$=\left(1+\frac{F_{n-1}}{F_n}\right)^2$$

$$\left(\frac{F_{n+1}}{F_n}\right)^2=1+2\left(\frac{F_{n-1}}{F_n}\right)+\left(\frac{F_{n-1}}{F_n}\right)^2.\ \blacksquare$$

Lemma 5. For every natural number n,

$$\left(\frac{F_{n+2}}{F_n}\right)^2=4+4\left(\frac{F_{n-1}}{F_n}\right)+\left(\frac{F_{n-1}}{F_n}\right)^2.$$

Proof.

$$\left(\frac{F_{n+2}}{F_n}\right)^2=\left(\frac{2F_n+F_{n-1}}{F_n}\right)^2$$

$$=\left(2+\frac{F_{n-1}}{F_n}\right)^2$$

$$=4+4\left(\frac{F_{n-1}}{F_n}\right)+\left(\frac{F_{n-1}}{F_n}\right)^2.\ \blacksquare$$

Result

In this section we give a proof of the new inequality which is our main result.

Theorem 6. (Main inequality)
For every natural number n,

$$2(F_n^4+F_{n+1}^4+F_{n+2}^4)\left(\frac{1}{F_n^2}+\frac{1}{F_{n+1}^2}+\frac{1}{F_{n+2}^2}\right)^2>100.$$

Proof.

$$2(F_n^4+F_{n+1}^4+F_{n+2}^4)\left(\frac{1}{F_n^2}+\frac{1}{F_{n+1}^2}+\frac{1}{F_{n+2}^2}\right)^2$$

$$=(F_n^2+F_{n+1}^2+F_{n+2}^2)^2\left(\frac{1}{F_n^2}+\frac{1}{F_{n+1}^2}+\frac{1}{F_{n+2}^2}\right)^2$$

(by Lemma 2.)

$$= \left(3 + \frac{F_{n+1}^2 + F_{n+2}^2}{F_n^2} + \frac{F_n^2 + F_{n+2}^2}{F_{n+1}^2} + \frac{F_n^2 + F_{n+1}^2}{F_{n+2}^2}\right)^2$$

$$= \left(3 + \left(\frac{F_{n+1}}{F_n}\right)^2 + \left(\frac{F_{n+2}}{F_n}\right)^2 + \left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_{n+2}}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2 + \left(\frac{F_{n+1}}{F_{n+2}}\right)^2\right)^2$$

$$= \left(9 + \frac{6F_{n-1}}{F_n} + \frac{2F_n}{F_{n+1}} + 2\left(\frac{F_{n-1}}{F_n}\right)^2 + 2\left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2 + \left(\frac{F_{n+1}}{F_{n+2}}\right)^2\right)^2$$

(by lemma 4 – 5.)

$$> \left(10 + \frac{6F_{n-1}}{F_n} + 2\left(\frac{F_{n-1}}{F_n}\right)^2 + 2\left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2 + \left(\frac{F_{n+1}}{F_{n+2}}\right)^2\right)^2$$

(by lemma 3.)

$$> 100. \blacksquare$$

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