1988

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S J. Dilworth
University of South Carolina - Columbia, dilworth@math.sc.edu

Publication Info
Published in *Houston Journal of Mathematics*, Volume 14, Issue 2, 1988, pages 179-190.

HOUSTON JOURNAL OF MATHEMATICS, Volume 14, No. 2, 1988.

INVOLUTIONS ON BANACH SPACES AND REFLEXIVITY

S. J. DILWORTH

1. Notation and results. Let E and F be (real or complex) Banach spaces. E is said to be finitely representable in F if, given $\epsilon > 0$ and a finite dimensional subspace E_0 of E, there exists a subspace F_0 of F such that $d(E_0, F_0) \leq 1 + \epsilon$, where

$$d(E_0, F_0) = \inf\{\|T\| \, \|T^{-1}\| : T \text{ is an isomorphism from } E_0 \text{ onto } F_0\}$$

denotes the Banach-Mazur distance coefficient. E is said to be super-reflexive if every Banach space which is finitely representable in E is reflexive. Super-reflexivity has been characterized in terms of the notion of J-convexity: suppose that $n \geq 1$ and that $\epsilon > 0$; E is said to be $J(n, \epsilon)$-convex if, for all $x_1, \ldots, x_n$ in the unit ball of E, we have

$$\inf_{1 \leq k \leq n-1} \|x_1 + \cdots + x_k - x_{k+1} - \cdots - x_n\| \leq n - \epsilon.$$

The "if" part of the following theorem was proved in [**12**] and [**5**], and the "only if" part was proved in [**10**].

THEOREM A. *E is super-reflexive if and only if E is $J(n, \epsilon)$-convex for some $n \geq 1$ and $\epsilon > 0$.*

The main purpose of this article is to extend Theorem A to a certain class of operators. To this end we introduce some new definitions: an operator T on E will be said to be $J(n, \epsilon)$-convexifying ($n \geq 1$ and $\epsilon > 0$) if, for all $x_1, \ldots, x_n$ in the unit ball of E, we have

$$\inf_{0 \leq k \leq n} \|x_1 + \cdots + x_k + T(x_{k+1} + \cdots + x_n)\| \leq n - \epsilon.$$

When no importance is placed on ϵ or n we shall say that T is $J(n)$-convexifying or simply J-convexifying. T will be said to be an involution (of order $n \geq 1$) if $T^n = I$, where I denotes the identity operator on E. The following main result is proved in Section 4 below.

THEOREM 1.1. *Suppose that E admits a J-convexifying involution. Then either c_0 is finitely representable in E or E is super-reflexive.*

Combining Theorem 1.1 with the "only if" part of Theorem A gives rise to the following characterization of super-reflexive Banach spaces.

THEOREM 1.2. *E is super-reflexive if and only if c_0 is not finitely representable in E and E admits a J-convexifying involution.*

Theorem 1.1 gives the following geometrical characterization of super-reflexive complex Banach spaces.

THEOREM 1.3. *Suppose that E is a complex Banach space, that $|\lambda| = 1$ and that $\lambda \neq 1$. Then E is super-reflexive if and only if there exist $n \geq 2$ and $\epsilon > 0$ such that for all $x_1, \ldots, x_n$ in the unit ball of E, we have*

$$\inf_{1 \leq k \leq n} \|x_1 + \cdots + x_k + \lambda x_{k+1} + \lambda x_n\| \leq n - \epsilon.$$

PROOF: Necessity is proved in Corollary 2.3 below. Sufficiency follows from Theorem 1.1 when λ is a root of unity (and so multiplication by λ is an involution) by observing that $c_0(C)$ does not satisfy the hypothesis. The case for general $\lambda \neq 1$ is simply a consequence of the density of the roots of unity in the unit circle.

It is not known to me whether the possibility of c_0 being finitely representable in E in the conclusion of Theorem 1.1 may be eliminated, but when E is a complex Banach space this can be done.

THEOREM 1.4. *Suppose that E is a complex Banach space. Then E is super-reflexive if and only if E admits a J-convexifying involution.*

PROOF: We need only prove sufficiency. It follows from the theory of algebraic operators (e.g., [**11**]) that if T is an involution on E, then E may be written as a direct sum of closed subspaces E_i on which T acts as multiplication by a root of unity. If $c_0(C)$ is finitely representable in E, then $c_0(C)$ is finitely representable in some E_i, but this means that T is not J-convexifying.

We conclude by stating a special case of Theorem 1.3 which may be regarded as a complex version of a theorem of R. C. James on uniformly non-square Banach spaces ([**8**]).

THEOREM 1.5. *Suppose that E is a complex Banach space, that $\epsilon > 0$, and that for all x, y in the unit ball of E, we have*

$$\min\{\|x+y\|, \|x+iy\|\} \leq 2-\epsilon.$$

Then E is reflexive.

2. J-convexifying operators. In this section we shall make use of the notion of the numerical range of an operator, which we now define. Suppose that E is a Banach space (either real or complex); the collection Π is defined by

$$\Pi = \{(x, f) : \|f\| = \|x\| = f(x) = 1\} \subset E \times E^*.$$

The numerical range of an operator T on E, denoted $W(T)$, is defined by

$$W(T) = \{f(Tx) : (x, f) \in \Pi\}.$$

PROPOSITION 2.1. *Suppose that T is a J-convexifying operator on E.*

(a) $\overline{W(T)} \subset \{z : \mathrm{Re}(z) < 1\}$.
(b) *If* $\|T\| = 1$ *then* $\|I+T\| < 2$.

PROOF: (a) Let $\epsilon > 0$ and $n \geq 1$ be given. If $\overline{W(T)} \not\subset \{z : \mathrm{Re}(z) < 1\}$ then there exists $(x, f) \in \Pi$ such that $\mathrm{Re}(f(Tx)) \geq 1 - \epsilon/2n$. It follows that for each $0 \leq k \leq n$, we have

$$\|kx + (n-k)Tx\| \geq |f(kx + (n-k)Tx)| \geq n - \frac{\epsilon}{2}.$$

Hence T is not J-convexifying, and the result follows.

(b) By a theorem of Lumer (e.g., [**2**, page 82]), we have

$$\sup\{\mathrm{Re}(z) : z \in W(T)\} = \lim_{\alpha \to 0} \frac{1}{\alpha}(\|I + \alpha T\| - 1);$$

but by (a) there exists $t < 1$ such that $\sup\{\mathrm{Re}(z) : z \in W(T)\} < t$, and so there exists $0 < \alpha < 1$ such that $\|I + \alpha T\| < 1 + \alpha t$. Since $\|T\| = 1$ it follows that $\|I + T\| < 2$.

The next result concerns J-convexifying operators on super-reflexive spaces. It generalizes one of the implications in Theorem A and serves as a partial converse to Proposition 2.1(b).

PROPOSITION 2.2. *Suppose that E is super-reflexive and that T is a norm one operator on E with $\|I+T\| < 2$. Then T is J-convexifying.*

PROOF: Select $\epsilon > 0$ and $\delta > 0$ such that $\|I+T\| + \epsilon + \delta < 2$. If T is not J-convexifying, then for each $n \geq 1$ there exist $x_1, \ldots, x_n$ in the unit ball of E such that

$$\inf_{0 \leq k \leq n} \|x_1 + \cdots + x_k + T(x_{k+1} + \cdots + x_n)\| > n - \epsilon.$$

Now suppose that $1 \leq k \leq n$, that $\alpha_i \geq 0$ and that $\sum_{i=1}^k \alpha_i = \sum_{i=k+1}^n \alpha_i = 1$. Then

$$\begin{aligned}
&\|\alpha_1 x_1 + \cdots + \alpha_k x_k + T(\alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n)\| \\
&\quad \geq \|x_1 + \cdots + x_k + T(x_{k+1} + \cdots + x_n)\| - \|(1-\alpha_1)x_1 \\
&\quad + \cdots + (1-\alpha_k)x_k + T((1-\alpha_{k+1})x_{k+1} + \cdots + (1-\alpha_n)x_n)\| \\
&\quad \geq n - \epsilon - \sum_{i=1}^k (1-\alpha_i) - \sum_{i=k+1}^n (1-\alpha_i)\|T\| \\
&\quad \geq n - \epsilon - (k-1) - (n-k-1) \\
&\quad = 2 - \epsilon.
\end{aligned}$$

Using the fact that $\|I+T\| + \epsilon + \delta < 2$, we obtain

$$\|\alpha_1 x_1 + \cdots + \alpha_k x_k - (\alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n)\| \geq 2 - \epsilon - \|I+T\| \geq \delta.$$

So for each $1 \leq k \leq n$, we have

$$d(\mathrm{conv}(x_1, \ldots, x_k), \mathrm{conv}(x_{k+1}, \ldots, x_n)) \geq \delta.$$

It follows from a characterization of super-reflexivity in [**9**] that E is not super-reflexive. This contradiction proves that T is J-convexifying.

The following immediate consequence of Proposition 2.2 completes the proof of Theorem 1.3.

COROLLARY 2.3. *Suppose that E is a super-reflexive complex Banach space, that $|\lambda| = 1$, and that $\lambda \neq 1$. Then there exist $n \geq 2$ and $\epsilon > 0$ such that for all $x_1, \ldots, x_n$ in the unit ball of E, we have*

$$\inf_{1 \leq k \leq n} \|x_1 + \cdots + x_k + \lambda x_{k+1} + \cdots + \lambda x_n\| \leq n - \epsilon.$$

When E is uniformly convex the converse of Proposition 2.1(a) is also true.

THEOREM 2.4. *Suppose that T is an operator on a uniformly convex space E. Then T is J-convexifying if and only if $\overline{W(T)} \subset \{z : \mathrm{Re}(z) < 1\}$.*

PROOF: Necessity is proved in Proposition 2.1 To prove the converse we shall suppose that $\overline{W(T)} \subset \{z : \mathrm{Re}(z) < k\}$, where $k < 1$. Select $\eta > 0$ so that $k + 2\eta\|T\| = m < 1$ and let $\epsilon = \eta^2/8$. Since E is uniformly convex there exists $\delta \in (0, \frac{1}{2}(1-m))$ such that if $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x+y\| \geq 2-\delta$, then $\|x-y\| \leq \epsilon$. Now select $n \geq 1$ such that $(\|T\| + \delta - 1)/(n-1) < \epsilon$. Suppose that $x_1, \ldots, x_n$ lie in the unit ball of E and that $\|x_1 + \cdots + x_n\| \geq n - \delta$. Then $\|x_i + x_j\| \geq 2 - \delta \quad (1 \leq i < j \leq n)$, and so $\|x_i - x_j\| \leq \epsilon$. To obtain a contradiction we shall suppose that $\|x_1 + \cdots + x_{n-1} + T(x_n)\| \geq n - \delta$. Select $f \in E^*$ such that $\|f\| = 1$ and $f(x_1 + \cdots + x_{n-1} + T(x_n)) = \|x_1 + \cdots + x_{n-1} + T(x_n)\|$. It follows that

$$\max_{1 \leq j \leq n} \mathrm{Re}(f(x_j)) \geq \frac{n - \delta - \|T\|}{n-1} \geq 1 - \epsilon,$$

and so $\mathrm{Re}(f(x_n)) \geq 1 - 2\epsilon \geq 1 - \eta^2/4$. By the Bishop-Phelps-Bollobas Theorem (e.g., [**2**]) there exists $(x, g) \in \Pi$ such that $\|x - x_n\| < \eta$ and $\|g - f\| < \eta$. So

$$\begin{aligned} \mathrm{Re}(f(Tx_n)) &\leq \mathrm{Re}(g(Tx_n)) + \eta\|T\| \\ &\leq \mathrm{Re}(g(Tx)) + 2\eta\|T\| \\ &\leq k + 2\eta\|T\| \\ &< 1 - 2\delta. \end{aligned}$$

Hence $\|x_1 + \cdots + x_{n-1} + Tx_n)\| = f(x_1 + \cdots + x_{n-1} + Tx_n) \leq n - 2\delta$, which is the desired contradiction. It follows that T is $J(n, \delta)$-convexifying.

COROLLARY 2.5. *The following are equivalent:*

(i) *E is super-reflexive;*

(ii) *if T is an operator on E such that $\overline{W(T)} \subset \{z : \mathrm{Re}(z) < 1\}$ then E can be renormed so that T is a J-convexifying operator with respect to the new norm.*

PROOF: (ii) implies (i) follows at once from Theorem A by considering $T = -I$. To prove that (i) implies (ii) we recall that a super-reflexive Banach space admits an equivalent uniformly convex norm ([**7**]). Moreover, it is well known (see e.g., [**1**, page 211]) that if $(E, \|\cdot\|)$ admits an equivalent uniformly convex norm then, given $\epsilon > 0$, there exists a uniformly convex norm $|||\cdot|||$ on E such that $(1-\epsilon)\|x\| \leq |||x||| \leq (1+\epsilon)\|x\|$ for all $x \in E$. A straightforward perturbation argument involving the Bishop-Phelps-Bollobas theorem proves that provided ϵ is sufficiently small the numerical range of T with respect to $||| \; |||$ still satisfies $\overline{W(T)} \subset \{z : \mathrm{Re}(z) < 1\}$. It now follows from Theorem 2.4 that T is J-convexifying with respect to $|||\cdot|||$.

We conclude this section by recalling the notion of an ultrapower of a Banach space, which will be needed in Section 4. Let F denote the collection of all bounded sequences $x = (x_n)_{n=1}^{\infty}$ in E, and let $\mathcal{U}$ be a non-trivial ultrafilter on N. A semi-norm on F is defined by $\|x\| = \lim_{\mathcal{U}} \|x_n\|$. Quotienting F by the kernel of this semi-norm and taking the completion gives rise to the ultrapower $E^N/\mathcal{U}$. An operator T on E induces an operator $\tilde{T}$ on $E^N/\mathcal{U}$ in the obvious way. The following proposition, whose straightforward proof is omitted, will be needed in Section 4.

LEMMA. *If T is a J-convexifying operator on E, then $\tilde{T}$ is J-convexifying on $E^N/\mathcal{U}$.*

We shall also need to use the fact that E is super-reflexive if and only if every ultrapower $E^N/\mathcal{U}$ is reflexive.

3. Generalization of the Brunel-Sucheston technique. To prove Theorem 1.1 we need to develop the machinery of the Brunel-Sucheston procedure ([**4**], [**5**]) in the more general setting of an algebra of operators acting on a normed space. Once the definitions have been decided upon much of the theory carries across from [**4**], [**5**] with only minor modifications; when this is so we shall merely state the corresponding result without proof.

Suppose that A is a real or complex algebra with identity and that E is a normed space. We shall say that E is an A-module if A acts as an algebra of bounded operators on E. Let $N = \{\alpha \in A : \alpha x = 0 \text{ for all } x \in E\}$; then A/N may and shall be regarded as a subalgebra of the algebra of all operators on E. Let S denote the space of all sequences $a = (a_i)_{i=1}^{\infty}$ of elements of A with only finitely many non-zero terms.

THEOREM 3.1. *Let $(y_n)_{n=1}^{\infty}$ be a bounded sequence in E and suppose that A/N is separable in the operator norm topology. There exists a semi-norm $\|\cdot\|$ on S and a subsequence $(x_n)_{n=1}^{\infty}$ such that, for all $a \in S$ and $\epsilon > 0$,* *there exists a positive integer ν such that*

$$\left| \left\| \sum_{i=1}^{\infty} a_i(x_{n_i}) \right\| - \|a\| \right| \leq \epsilon$$

for all integers $\nu \leq n_1 < n_2 < \cdots$.

We shall assume throughout the remainder of this section that $(x_n)_1^\infty$, $(y_n)_1^\infty$ and the seminorm $\|\cdot\|$ are fixed. If $\tilde{K}$ is the kernel of this semi-norm then $S/\tilde{K}$ is itself a normed A-module with the action defined coordinatewise.

PROPOSITION 3.2. *$S/\tilde{K}$ is finitely representable in E.*

We now introduce a type of finite representability which appropriately reflects the A-module structure. Suppose that E and F are normed A-modules. Then E will be said to be A-finitely representable in F if, for all positive integers n and N, for all $z_1, \ldots, z_n$ in E, and for all n-tuples $(\alpha_1^k, \ldots, \alpha_n^k)$ $(1 \leq k \leq N)$ of elements of A, there exist $w_1, \ldots, w_n$ in F such that

$$\left| \left\| \sum_{i=1}^{n} \alpha_i^k z_i \right\| - \left\| \sum_{i=1}^{n} \alpha_i^k w_i \right\| \right| < \epsilon \quad (1 \leq k \leq N).$$

A standard compactness argument shows that the above definition coincides with the usual notion of finite representability when E and F are just normed spaces.

PROPOSITION 3.3. *$S/\tilde{K}$ is A-finitely representable in E.*

PROOF: Suppose that $\hat{z}_1, \ldots, \hat{z}_n$ are any vectors in $S/\tilde{K}$ and that $(\alpha_1^k, \ldots, \alpha_n^k)$ $(1 \leq k \leq N)$ are n-tuples of elements of A. Let $z_1, \ldots, z_n$ be representatives from S of $\hat{z}_1, \ldots, \hat{z}_n$. For $m \geq 1$, let $R_m : S \to E$ be the A-module homomorphism uniquely defined by $R_m(e_k) = x_{m+k}$ (here $(e_k)_{k=1}^\infty$ is the canonical basis of S as a free A-module). Given $\epsilon > 0$, there exists $m \geq 1$ such that

$$\left| \left\| \sum_{i=1}^{n} \alpha_i^k z_i \right\| - \left\| R_m(\sum_{i=1}^{n} \alpha_i^k z_i) \right\| \right| < \epsilon$$

for each $1 \leq k \leq N$. Setting $w_i = R_m(z_i)$ and using the fact that R_m is an A-module homomorphism, we obtain

$$\left| \left\| \sum_{i=1}^{n} \alpha_i^k \hat{z}_i \right\| - \left\| \sum_{i=1}^{n} \alpha_i^k w_i \right\| \right| < \epsilon$$

for each $1 \leq k \leq N$. This completes the proof.

$(S, \|\cdot\|)$ has the property that, for all $k \geq 1$, for all natural numbers $n_1 < n_2 < \cdots < n_k$, and for all $a = \sum_{i=1}^{k} a_i e_i$ in S, we have $\|\sum_{i=1}^{k} a_i e_i\| = \|\sum_{i=1}^{k} a_i e_{n_i}\|$. In accordance with [4] such a semi-norm on S will be called "invariant under spreading" (or I.S.). We turn now to define an analogue of the "equal signs additive" norm of [5], [6]. For each $n \geq 1$, the averaging operator $A_n : S \to S$ is defined by $A_n(e_k) = \frac{1}{n}(e_k + e_{k+1} + \cdots + e_{k+n-1})$ with extension to S by A-linearity. Given $a = a_1 e_1 + \cdots + a_r e_r$ in S, we consider the vector $a_1 A_{n_1}(e_{s_1}) + \cdots + a_r A_{n_r}(e_{s_r})$, where $s_1 > 0$, $s_2 \geq s_1 + n_1, \ldots, s_r \geq s_{r-1} + n_{r-1}$. The I.S. property of $\|\cdot\|$ guarantees that the semi-norm of this vector does not depend on the choice of $s_1, \ldots, s_r$; it shall be denoted by $F(a; n_1, \ldots, n_r)$.

Proposition 3.4. *For each $a = a_1 e_1 + \cdots + a_r e_r$ in S, the limit of $F(a; n_1, \ldots, n_r)$ as $\inf\{n_i : 1 \leq i \leq r\} \to \infty$ exists. This limit, denoted $|||a|||$, is a semi-norm on S.*

If K denotes the kernel of $|||\cdot|||$, then S/K is a normed A-module; exactly as in Proposition 3.2 we have the following.

Proposition 3.5. *$(S/K, |||\cdot|||)$ is both finitely representable and A-finitely representable in E.*

Let $|\cdot|$ be a semi-norm on S. Then $|\cdot|$ will be said to be "equal terms additive" (E.T.A.) if, for each $a = a_1 e_1 + \cdots + a_r e_r$ in S with $a_i = a_{i+1}$ for some $1 \leq i \leq r-1$, we have

$$|a| = |a_1 e_1 + \cdots + a_{i-1} e_{i-1} + (a_i + a_{i+1}) e_i + a_{i+2} e_{i+2} + \cdots + a_r e_r|.$$

It is easily seen that an E.T.A. semi-norm is automatically I.S.

Proposition 3.6: $|||\cdot|||$ is an E.T.A. semi-norm on S.

Proof: Suppose that $a = a_1 e_1 + \cdots + a_r e_r$ with $a_i = a_{i+1}$. Let $b = a_1 e_1 + \cdots + a_{i-1} e_{i-1} + (a_i + a_{i+1}) e_i + a_{i+2} e_{i+2} + \cdots + a_r e_r$. The I.S. property implies that $F(a; n_1, \ldots, n_{i-1} N, N, n_{i+2}, \ldots, n_r) = F(b; n_1, \ldots, n_{i-1}, 2N, n,$

$n_{i+2}, \ldots, n_r)$ for all $n_1, \ldots, n_{i-1}, n_{i+2}, \ldots, n_r, N$ and n. The result follows at once.

4. *J*-convexifying involutions.

This section is devoted to a proof of Theorem 1.1. We shall use the ideas of the previous section and follow the strategy of the proof of Theorem A given in [5]. Anything from [5] which transfers with only minor alteration will be stated without proof. Using the notation of the previous section we shall show that the sequence $(x_n)_{n=1}^{\infty}$ contains a subsequence which is convergent in Cesaro mean when the hypotheses of Theorem 1.1 are met. This will show that E has the Banach-Saks property and, in particular, that E is reflexive.

PROPOSITION 4.1 ([5]): If $|||e_1 - e_2||| = 0$ then $(x_n)_{n=1}^{\infty}$ contains a subsequence which is convergent in Cesaro mean.

LEMMA 4.2. *Suppose that T is a J-convexifying involution on a Banach space E. Then there exists $k \geq 1$ such that $I + T + T^2 + \cdots + T^k = 0$.*

PROOF: Suppose that T is an involution of order $k+1$. Then $\|T^n\|^{1/n} \to 1$ as $n \to \infty$, and so $T - \lambda I$ is invertible for all $\lambda > 1$. It follows that either $T - I$ is invertible or that $T - I$ fails to be an isomorphism onto its range: in the complex case this is just the familiar fact that every point in the boundary of the spectrum of T is an approximate eigenvalue. If $T - I$ is not an isomorphism onto its range then, given $\epsilon > 0$, there exists a unit vector x in E with $\|Tx - x\| < \epsilon$. It follows that, for each $n \geq 1$, we have

$$\inf_{0 \leq k \leq n} \|kx + T((n-k)x)\| \geq n - n\epsilon.$$

Since ϵ is arbitrary, this contradicts the fact that T is J-convexifying. So $T - I$ is invertible and hence $I + T + T^2 + \cdots + T^k = 0$.

Now suppose that the element α of A satisfies $I + \alpha + \cdots + \alpha^{k-1} = 0$. We shall say that α is cyclic of order k. A sequence of vectors $(f_n)_{n=1}^{\infty}$ in S is defined by

$$f_n = e_{(n-1)k+1} + \alpha e_{(n-1)k+2} + \cdots + \alpha^{k-1} e_{nk};$$

the real vector subspace spanned by $(f_n)_{n=1}^{\infty}$ will be denoted F.

PROPOSITION 4.3. *If* $|||e_1 - e_2||| \neq 0$ *then* $(F, |||\ |||)$ *is a normed space.*

PROPOSITION 4.4. $(f_n)_{n=1}^{\infty}$ *is an orthogonal sequence in* F.

PROOF: We have to show that, for all $m \geq 1$, for all real $\lambda_1, \ldots, \lambda_m$, and for each $1 \leq r \leq m$, we have

$$|||\lambda_1 f_1 + \cdots + \lambda_m f_m||| \geq |||\lambda_1 f_1 + \cdots + \lambda_{r-1} f_{r-1} + \lambda_{r+1} f_{r+1} + \cdots + \lambda_m f_m|||.$$

The I.S. property is used to write the expression on the left hand side as each of the following n expressions:

$$|||y + \lambda_r(e_{(r-1)k+1} + \alpha e_{(r-1)k+2} + \cdots + \alpha^{k-1} e_{rk}) + z|||;$$

$$|||y + \lambda_r(e_{(r-1)k+2} + \alpha e_{(r-1)k+3} + \cdots + \alpha^{k-1} e_{rk+1}) + z|||;$$

down to

$$|||y + \lambda_r(e_{(r-1)k+n} + \alpha e_{(r-1)k+n+1} + \cdots + \alpha^{k-1} e_{rk+n-1}) + z|||,$$

where $y = \sum_{i=1}^{r-1} \lambda_i f_i$ and $z = U_n(\sum_{i=r+1}^{m} \lambda_i f_i)$ (here $U_n : S \to S$ is the A-module homomorphism defined by $U_n(e_k) = e_{n+k}$ for all $k \geq 1$). Taking the average of these, and using the triangle inequality and the fact that $I + \alpha \cdots + \alpha^{k-1} = 0$, we obtain

$$|||\lambda_1 f_1 + \cdots + \lambda_m f_m||| \geq |||y + z||| - \frac{|\lambda_r|}{n} |||a + b|||,$$

where

$$a = e_{(r-1)k+1} + (1 + \alpha) e_{(r-1)k+2} + \cdots + (1 + \alpha + \cdots + \alpha^{k-2}) e_(r-1)k + k - 1$$

and

$$b = \alpha^{k-1} e_{rk+n-1} + (\alpha^{k-1} + \alpha^{k-2}) e_{rk+n-2} + \cdots + (\alpha^{k-1} + \cdots + \alpha) e_{rk+n-k+1}.$$

Using the I.S. property and taking the limit as n tends to infinity gives the required result.

COROLLARY 4.5. (a) $(f_n)_{n=1}^{\infty}$ *is an unconditional basic sequence in* F.

(b) *Either* c_0 *is finitely representable in* F *or* $|||f_1 + \cdots + f_n|||$ *increases to infinity with* n.

PROPOSITION 4.6. *Suppose that* E *is a normed* A*-module and that* α *is a* J*-convexifying cyclic element of* A. *Then either* c_0 *is finitely representable in* E *or* E *is reflexive.*

PROOF: If E is not reflexive then there exists a bounded sequence $(y_n)_{y=1}^{\infty}$ in E which has no Cesaro mean convergent subsequence. It follows from Propositions 4.1 and 4.3 that the space F constructed above is a normed space. If c_0 is not finitely representable in E, then by Proposition 3.5 c_0 is not finitely representable in S/K, of which F is a subspace; so by Corollary 4.5(b), $|||f_1+\cdots+f_n|||$ increases to infinity with n. Now suppose that α is cyclic of order k and is $J(r,\epsilon)$-convexifying. For each $1 \le j \le r$ and $n \ge 1$, we define

$$\begin{aligned} v_n^j =&(e_{j+r} + \alpha e_{j+2r} + \cdots + \alpha^{k-1} e_{j+kr}) \\ &\qquad + \cdots + (e_{j+(n-1)kr+r} + \cdots + \alpha^{k-1} e_{j+nkr}). \end{aligned}$$

Let $d_n^s = v_n^1 + \cdots + v_n^s + \alpha(v_n^{s+1} + \cdots + v_n^r)$ $(0 \le s \le r)$; we write $d_n^s = S_1 + S_2 + S_3$ by grouping the terms as follows:

$$\begin{aligned} S_1 &= -e_{s+1} - \cdots - e_r; \\ S_2 &= \{(e_{s+1} + \cdots + e_{s+r}) + \alpha(e_{r+s+1} + \cdots + e_{2r+s}) \\ &\quad + \alpha^2(e_{2r+s+1} \\ &\quad + \cdots + e_{3r+s}) + \cdots + \alpha^{k-1}(e_{s+(k-1)r+1} + \cdots + e_{s+kr})\} \\ &\quad + \cdots + \{(e_{s+1+(n-1)kr} + \cdots + e_{s+((n-1)k+1)r}) \\ &\quad + \cdots + \alpha^{k-1}(e_{s+1+(nk-1)r} + \cdots + e_{nkr+s})\}; \\ S_3 &= e_{nkr+s+1} + \cdots + e_{(nk+1)r}. \end{aligned}$$

The I.S. property implies that

$$|||v_n^j||| = |||f_1 + \cdots + f_n||| \qquad (1 \ge j \ge r),$$

and so $|||S_1|||/|||v_n^j|||$ and $|||S_3|||/|||v_n^j|||$ both tend to zero as n tends to infinity. Moreover, the E.T.A. property implies that $|||S_2||| = r|||v_n^j|||$. Let $z_n^j = v_n^j/|||v_n^j|||$, so that $|||z_n^j||| = 1$. Then

$$\inf_{0 \le j \le r} |||z_n^1 + \cdots + z_n^j + \alpha(z_n^{j+1} + \cdots + z_n^r)||| \ge r - \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$. Hence α is not $J(r)$-convexifying for $(S/K, ||| \ |||)$. But by Proposition 3.5 S/K is A-finitely representable in E, and so α is not $J(r)$-convexifying for E. This contradiction completes the proof of the proposition.

PROOF OF THEOREM 1.1: Let T be a J-convexifying involution on a Banach space E and suppose that c_0 is not finitely representable in E. By Lemma 2.6 the induced operator $\tilde{T}$ on the ultrapower $E^N/\mathcal{U}$ is J-convexifying; moreover, $\tilde{T}$ is clearly also an involution. Let A be the subalgebra generated by $\tilde{T}$ of the algebra of all bounded operators on $E^N/\mathcal{U}$. Then $E^N/\mathcal{U}$ is a normed A-module, and by Lemma 4.2 $\tilde{T}$ is a J-convexifying cyclic element of A. It follows from Proposition 4.6 that $E^N/\mathcal{U}$ is reflexive, and so E is super-reflexive.

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University of Missouri-Columbia
Columbia, Missouri 65211

Received July 30, 1985