12-1972

Stochastic Integro-Differential Equations of Volterra Type

William J. Padgett
University of South Carolina - Columbia, padgettw@bellsouth.net

Chris P. Tsokos

Publication Info
Siam Journal on Applied Mathematics, Volume 23, Issue 4, 1972, pages 499-512.

SIAM J. APPL. MATH.
Vol. 23, No. 4, December 1972

ON A STOCHASTIC INTEGRO-DIFFERENTIAL EQUATION OF VOLTERRA TYPE*

W. J. PADGETT† AND CHRIS P. TSOKOS‡

Abstract. A nonlinear stochastic integro-differential equation of the form

$$x'(t;\omega) = h(t, x(t;\omega)) + \int_0^t k(t,\tau;\omega)f(x(\tau;\omega))\,d\tau,$$

where $t \geqq 0$ ($' = d/dt$), and $\omega \in \Omega$, the supporting set of a complete probability measure space (Ω, A, P), is studied with respect to the existence of a unique random solution. Results are also given concerning the statistical behavior of the random solution as $t \to \infty$, and an application to differential systems with random parameters is presented.

1. Introduction. Many investigations have been carried out concerning the existence, uniqueness, and asymptotic properties of solutions of nonstochastic integro-differential equations of Volterra type [6]–[11], [13]–[14], [17], to mention a few. However, in many applications, such as problems in reactor dynamics [7], due to the complex random nature of the situation, the phenomenon being studied should be more realistically considered in a *stochastic* framework, resulting in a *stochastic* integro-differential equation. Therefore, in this paper we shall be concerned with extending some of the deterministic results (for example, results in [8], [10], [14], [17]) to the more general stochastic setting. That is, we shall consider a nonlinear stochastic integro-differential equation of Volterra type of the form

$$x'(t;\omega) = h(t, x(t;\omega)) + \int_0^t k(t,\tau;\omega)f(x(\tau;\omega))\,d\tau, \tag{1.1}$$

where $t \in R_+ = [0, \infty)$ ($' = d/dt$), and

(i) $\omega \in \Omega$, the supporting set of a probability measure space (Ω, A, P);
(ii) $x(t;\omega)$ is the unknown stochastic process for $t \in R_+$;
(iii) $h(t, x)$ is a scalar function of $t \in R_+$ and scalar x;
(iv) $k(t,\tau;\omega)$ is the *stochastic kernel* defined for t and τ satisfying $0 \leqq \tau \leqq t < \infty$; and
(v) $f(x)$ is a scalar function of x.

Stochastic or random integral equations have been studied extensively by Anderson [1], Bharucha-Reid [2]–[4], Padgett and Tsokos [16], to mention a few. To our knowledge, however, there has been no work done concerning nonlinear stochastic integro-differential equations of the form (1.1).

The purpose of this paper is to obtain very general conditions concerning the stochastic processes in equation (1.1) which guarantee the existence and uniqueness of a *random solution* $x(t;\omega)$ and to investigate the asymptotic *statistical* behavior of such a random solution. In addition, the usefulness of the results will be illus-

* Received by the editors August 3, 1971, and in revised form March 27, 1972.

† Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208.

‡ Department of Statistics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061 and Department of Mathematics, University of South Florida, Tampa, Florida 33620.

trated with an application to stochastic differential systems. To accomplish these objectives we shall utilize the spaces of functions and admissibility theory which were introduced into the study of stochastic integral equations quite recently by Tsokos [18]. The theory of admissibility has been used in the case of nonstochastic integral equations by Corduneanu [5], among others.

The importance of stochastic integro-differential equations of the form (1.1) lies in the fact that they arise in many situations. For example, equations of this kind occur in the stochastic formulation of problems in reactor dynamics [7], in the study of the growth of biological populations [11], in the theory of automatic systems resulting in delay-differential equations, Oğuztöreli [15], and in many other problems occurring in the general areas of biology, physics, and engineering.

In § 2, we shall define the spaces of functions which will be used, state the assumptions concerning the random functions in (1.1), and give some necessary lemmas. The main results will be presented in § 3. Section 4 will contain some results concerning the asymptotic behavior (as $t \to \infty$) of the random solution, and § 5 will contain an application of the results to differential systems with random parameters.

2. Preliminaries. With respect to the functions in the stochastic integro-differential equation (1.1) we make the following assumptions: The random solution $x(t;\omega)$ will be considered as a function of $t \in R_+$ with values in the space $L_2(\Omega, A, P)$. That is, $x(t;\omega)$ is a second order stochastic process defined on R_+. The function $h(t, x(t;\omega))$, called the *stochastic perturbing term*, under certain conditions will also be a function of t with values in $L_2(\Omega, A, P)$, and $f(x(t;\omega))$ will be considered as a function from R_+ into $L_2(\Omega, A, P)$, similarly.

With respect to the stochastic kernel, we assume that, for each t and τ such that $0 \leqq \tau \leqq t < \infty$, $k(t, \tau;\omega)$ is essentially bounded. That is, for each t and τ satisfying $0 \leqq \tau \leqq t < \infty$, $k(t,\tau;\omega) \in L_\infty(\Omega, A, P)$. The norm of $k(t,\tau;\omega)$ in $L_\infty(\Omega, A, P)$ will be denoted by

$$|||k(t,\tau;\omega)||| = P\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega} |k(t,\tau;\omega)|.$$

Also, the mapping

$$(t,\tau) \to k(t,\tau;\omega)$$

from the set

$$\Delta = \{(t,\tau) : 0 \leqq \tau \leqq t < \infty\}$$

into $L_\infty(\Omega, A, P)$ is continuous, and further, whenever $\tau_n \to \tau$ as $n \to \infty$,

$$|||k(s,\tau_n;\omega) - k(s,\tau;\omega)||| \to 0$$

as $n \to \infty$ uniformly in s for $0 \leqq s \leqq t$, $t \in R_+$. It will be assumed also that for each fixed t and τ,

$$|||k(s,\tau;\omega)||| \leqq M_{(t,\tau)} \quad \text{uniformly for } \tau \leqq s \leqq t,$$

where $M_{(t,\tau)} > 0$ is some constant depending only on t and τ, $0 \leqq \tau \leqq t < \infty$.

DEFINITION 2.1. We define the space $C_c = C_c(R_+, L_2(\Omega, A, P))$ to be the space of all continuous functions from R_+ into $L_2(\Omega, A, P)$ with the topology of uniform convergence on every interval $[0, Q], Q > 0$. That is, $x_n(t;\omega) \to x(t;\omega)$ in $C_c(R_+, L_2(\Omega, A, P))$ if and only if

$$\lim_{n\to\infty} \|x_n(t;\omega) - x(t;\omega)\|_{L_2(\Omega,A,P)} = 0$$

uniformly on every interval $[0, Q]$, $Q > 0$.

We denote the norm in $L_2(\Omega, A, P)$ by

$$\|x(t;\omega)\| = \|x(t;\omega)\|_{L_2(\Omega,A,P)} = \left\{\int_\Omega |x(t;\omega)|^2 \, dP(\omega)\right\}^{1/2}$$

for each $t \in R_+$.

Note that $C_c(R_+, L_2(\Omega, A, P))$ is a Fréchet space with metric defined by the Fréchet combination of the following family of seminorms [20]:

$$\|x(t;\omega)\|_n = \sup_{0\leqq t\leqq n} \|x(t;\omega)\|, \qquad n = 1, 2, \cdots.$$

(Also, $C_c(R_+, L_2(\Omega, A, P))$ is a locally convex space whose topology is defined by the above family of seminorms [20, pp. 24–26].)

DEFINITION 2.2. We denote by C_g the Banach space of all continuous functions from R_+ into $L_2(\Omega, A, P)$ such that there exist a constant $M > 0$ and a positive continuous function $g(t)$ on R_+ satisfying

$$\|x(t;\omega)\| \leqq Mg(t), \quad t \in R_+.$$

The norm in $C_g(R_+, L_2(\Omega, A, P))$ is defined by

$$\|x(t;\omega)\|_{C_g} = \sup_{t\geqq 0}\left\{\frac{\|x(t;\omega)\|}{g(t)}\right\}.$$

Note that $C_g \subset C_c$ and that for $g(t) = 1$ for all $t \in R_+$ we have C_1, the space of all *bounded* continuous functions from R_+ into $L_2(\Omega, A, P)$.

Let B, $D \subset C_c(R_+, L_2(\Omega, A, P))$ be Banach spaces, and let T be a linear operator from $C_c(R_+, L_2(\Omega, A, P))$ into itself. We give the following definitions with respect to B, D, and T.

DEFINITION 2.3. The pair of Banach spaces (B, D) is said to be *admissible* with respect to the linear operator T if and only if $T(B) \subset D$.

DEFINITION 2.4. The Banach space B is *stronger* than $C_c(R_+, L_2(\Omega, A, P))$ if every convergent sequence in B also converges in C_c, but the converse is not true in general.

DEFINITION 2.5. We call $x(t;\omega)$ a *random solution* of the equation (1.1) if it satisfies the equation P-a.e. and is a second order stochastic process on R_+

We now state the following lemma which is given by Tsokos [18].

LEMMA 2.1. *Let T be a continuous linear operator from $C_c(R_+, L_2(\Omega, A, P))$ into itself. If B and D are Banach spaces stronger than C_c and if (B, D) is admissible with respect to T, then T is a continuous linear operator from B into D.*

The proof follows easily from the closed-graph theorem.

If the operator $T: B \to D$ is continuous, then it is bounded, and there exists a constant $K_0 > 0$ such that

$$\|(Tx)(t;\omega)\|_D \leqq K_0 \|x(t;\omega)\|_B.$$

Define the integral operators T_1 and T_2 on $C_c(R_+, L_2(\Omega, A, P))$ as follows:

$$(T_1x)(t;\omega) = \int_0^t x(\tau;\omega)\,d\tau \tag{2.1}$$

and

$$(T_2x)(t;\omega) = \int_0^t K(t,\tau;\omega)x(\tau;\omega)\,d\tau, \tag{2.2}$$

where

$$K(t,\tau;\omega) = \int_\tau^t k(s,\tau;\omega)\,ds. \tag{2.3}$$

These integral operators will be needed in obtaining the existence and uniqueness of a random solution of (1.1). We shall prove two lemmas concerning the continuity of T_1 and T_2 as mappings from $C_c(R_+, L_2(\Omega, A, P))$ into itself.

LEMMA 2.2. *The operator T_1 defined by* (2.1) *is a continuous mapping from $C_c(R_+, L_2(\Omega, A, P))$ into itself.*

Proof. For $x(t;\omega) \in C_c$, it follows that

$$\|(T_1x)(t;\omega)\| \leqq \int_0^t \|x(\tau;\omega)\|\,d\tau \leqq M_t t < \infty$$

for each $t \in R_+$, since $\|x(\tau;\omega)\|$ is a continuous function on the interval $[0, t]$ and hence bounded by some M_t. Thus, $(T_1x)(t;\omega) \in L_2(\Omega, A, P)$ for each $t \in R_+$.

Let $t_n \to t$ as $n \to \infty$. Then

$$\begin{aligned}\|(T_1x)(t_n;\omega) - (T_1x)(t;\omega)\| &\leqq \int_{t_n}^t x(\tau;\omega)\,d\tau \\ &\leqq \int_{t_n}^t \|x(\tau;\omega)\|\,d\tau.\end{aligned}$$

So for $\varepsilon > 0$ there exists an N_1 such that $n > N_1$ implies that the right-hand side of the last inequality is less than ε. Hence, T_1x is a continuous function from R_+ into $L_2(\Omega, A, P)$, that is, continuous in mean square on R_+, so that T_1 maps from C_c into itself.

To show that $T_1: C_c \to C_c$ is continuous, let $x_n(t;\omega) \to x(t;\omega)$ in C_c as $n \to \infty$. Then

$$\begin{aligned}\|(T_1x_n)(t;\omega) - (T_1x)(t;\omega)\| &\leqq \int_0^t \|x_n(\tau;\omega) - x(\tau;\omega)\|\,d\tau \\ &\leqq \int_0^Q \|x_n(\tau;\omega) - x(\tau;\omega)\|\,d\tau\end{aligned}$$

for $t \in [0, Q]$. But by definition, as $n \to \infty$,

$$\|x_n(\tau;\omega) - x(\tau;\omega)\| \to 0$$

uniformly in $[0, Q]$, $Q > 0$. Therefore, for $\varepsilon > 0$ there exists an N_2 such that $n > N_2$ implies that

$$\|(T_1 x_n)(t;\omega) - (T_1 x)(t;\omega)\| < \varepsilon Q,$$

completing the proof.

LEMMA 2.3. *The operator T_2 defined by equation* (2.2) *is a continuous mapping from $C_c(R_+, L_2(\Omega, A, P))$ into itself.*

Proof. We must first show that the function $K(t, \tau;\omega)$ given by (2.3) is in the space $L_\infty(\Omega, A, P)$ and is a continuous mapping from the set Δ into $L_\infty(\Omega, A, P)$. For fixed t and τ satisfying $0 \leqq \tau \leqq t < \infty$, we have that

$$|||K(t,\tau;\omega)||| \leqq \int_\tau^t |||k(s,\tau;\omega)|||\, ds \leqq M_{(t,\tau)}(t-\tau)$$

by the assumptions on $k(s, \tau;\omega)$. Hence, $K(t, \tau;\omega) \in L_\infty(\Omega, A, P)$ for each t and τ. Suppose $(t_n, \tau_n) \to (t, \tau)$ as $n \to \infty$. Then

$$\begin{aligned}
&|||K(t_n,\tau_n;\omega) - K(t,\tau;\omega)|||\\
&\leqq \left|\left|\left| \int_{\tau_n}^{t_n} k(s,\tau_n;\omega)\,ds - \int_\tau^t k(s,\tau;\omega)\,ds \right|\right|\right|\\
&= \left|\left|\left| \left[\int_{\tau_n}^{t_n} k(s,\tau_n;\omega)\,ds - \int_{\tau_n}^t k(s,\tau_n;\omega)\,ds\right]\right.\right.\right.\\
&\quad \left.\left.\left. + \left[\int_{\tau_n}^t k(s,\tau_n;\omega)\,ds - \int_{\tau_n}^t k(s,\tau;\omega)\,ds\right] + \left[\int_{\tau_n}^t k(s,\tau;\omega)\,ds - \int_\tau^t k(s,\tau;\omega)\,ds\right]\right|\right|\right|\\
&\leqq \left|\left|\left| \int_{t_n}^t k(s,\tau_n;\omega)\,ds \right|\right|\right|\\
&\quad + \left|\left|\left| \int_{\tau_n}^t [k(s,\tau_n;\omega) - k(s,\tau;\omega)]\,ds \right|\right|\right| + \left|\left|\left| \int_{\tau_n}^\tau k(s,\tau;\omega)\,ds \right|\right|\right|.
\end{aligned}$$

For $\varepsilon > 0$ there exists an N_1 such that $n > N_1$ implies (since $|||k(t,\tau;\omega)|||$ is continuous in τ)

$$\left|\left|\left| \int_{t_n}^t k(s,\tau_n;\omega)\,ds \right|\right|\right| \leqq \int_{t_n}^t |||k(s,\tau_n;\omega)|||\,ds < \frac{\varepsilon}{3}.$$

Also, there exist an N_2 and a $\delta > 0$ such that $n > N_2$ implies that $|\tau - \tau_n| < \delta$ and

$$|||k(s,\tau_n;\omega) - k(s,\tau;\omega)||| < \frac{\varepsilon}{3(t-\tau+\delta)}$$

uniformly in s for $\tau_n \leqq s \leqq t$ by the conditions on k so that

$$\begin{aligned}
\left|\left|\left| \int_{\tau_n}^t [k(s,\tau_n;\omega) - k(s,\tau;\omega)]\,ds \right|\right|\right| &\leqq \int_{\tau_n}^t |||k(s,\tau_n;\omega) - k(s,\tau;\omega)|||\,ds\\
&< \frac{\varepsilon}{3(t-\tau+\delta)}(t-\tau+\tau-\tau_n) < \frac{\varepsilon}{3(t-\tau+\delta)}(t-\tau+\delta)\\
&= \varepsilon/3.
\end{aligned}$$

Likewise, there exists an N_3 such that $n > N_3$ implies

$$\left|\left|\left| \int_{\tau_n}^{\tau} k(s, \tau; \omega)\, ds \right|\right|\right| < \frac{\varepsilon}{3}.$$

Hence, for $\varepsilon > 0$ there exists an $N = \max\{N_1, N_2, N_3\}$ so that for $n > N$ we have

$$|||K(t_n, \tau_n; \omega) - K(t, \tau; \omega)||| < \varepsilon,$$

that is, the mapping $K\colon \Delta \to L_\infty(\Omega, A, P)$ is continuous.

Now, since $K(t, \tau; \omega) \in L_\infty(\Omega, A, P)$ for each t and τ, $0 \leqq \tau \leqq t < \infty$, we have that for each $x(t; \omega) \in C_c$, the product $K(t, \tau; \omega)x(\tau; \omega)$ is in $L_2(\Omega, A, P)$. Thus,

$$\|(T_2x)(t; \omega)\| \leqq \int_0^t |||K(t, \tau; \omega)||| \cdot \|x(\tau; \omega)\|\, d\tau < \infty,$$

since $|||K(t, \tau; \omega)|||$ and $\|x(\tau; \omega)\|$ are continuous in τ on the interval $[0, t]$ and are therefore bounded on $[0, t]$, and so $(T_2x)(t; \omega)$ is in $L_2(\Omega, A, P)$.

Let $t_n \to t$ as $n \to \infty$. By the continuity condition on $K(t, \tau; \omega)$, we obtain

$$\begin{aligned}
&\|(T_2x)(t_n; \omega) - (T_2x)(t; \omega)\| \\
&= \left\| \int_0^{t_n} K(t_n, \tau; \omega)x(\tau; \omega)\, d\tau - \int_0^t K(t, \tau; \omega)x(\tau; \omega)\, d\tau \right\| \\
&\leqq \left\| \int_0^{t_n} K(t_n, \tau; \omega)x(\tau; \omega)\, d\tau - \int_0^t K(t_n, \tau; \omega)x(\tau; \omega)\, d\tau \right\| \\
&\quad + \left\| \int_0^{t} K(t_n, \tau; \omega)x(\tau; \omega)\, d\tau - \int_0^t K(t, \tau; \omega)x(\tau; \omega)\, d\tau \right\| \\
&\leqq \left\| \int_{t_n}^{t} K(t_n, \tau; \omega)x(\tau; \omega)\, d\tau \right\| \\
&\quad + \int_0^t |||K(t_n, \tau; \omega) - K(t, \tau; \omega)||| \cdot \|x(\tau; \omega)\|\, d\tau \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Thus, $(T_2x)(t; \omega)$ is continuous in the mean square.

To show that $T_2 : C_c \to C_c$ is continuous, let $x_n(t; \omega) \to x(t; \omega)$ in $C_c(R_+, L_2(\Omega, A, P))$ as $n \to \infty$. Then

$$\begin{aligned}
\|(T_2x_n)(t; \omega) - (T_2x)(t; \omega)\| &\leqq \int_0^t |||K(t, \tau; \omega)||| \cdot \|x_n(\tau; \omega) - x(\tau; \omega)\|\, d\tau \\
&\leqq \int_0^Q |||K(t, \tau; \omega)||| \cdot \|x_n(\tau; \omega) - x(\tau; \omega)\|\, d\tau,
\end{aligned}$$

where $Q \geqq t$. Since $|||K(t, \tau; \omega)|||$ is continuous in (t, τ), it is bounded by some $M_Q > 0$ in the compact region $\{(t, \tau): 0 \leqq t \leqq Q,\ 0 \leqq \tau \leqq Q\}$. By definition, we have for $\varepsilon > 0$ that there exists an N so that for $n > N$,

$$\|x_n(\tau; \omega) - x(\tau; \omega)\| < \varepsilon$$

uniformly in $[0, Q]$, and hence for $n > N$,

$$\|(T_2 x_n)(t;\omega) - (T_2 x)(t;\omega)\| < \varepsilon M_Q Q$$

for all $t \in [0, Q]$, where $Q > 0$ is arbitrary. Therefore, $(T_2 x_n)(t;\omega) \to (T_2 x)(t;\omega)$ in C_c as $n \to \infty$, that is, $T_2 : C_c \to C_c$ is continuous, completing the proof.

DEFINITION 2.6. The random solution $x(t;\omega)$ is *stochastically exponentially stable* if there exist constants $\alpha > 0$ and $\beta > 0$ such that for each $t \geqq 0$,

$$\|x(t;\omega)\| \leqq \beta e^{-\alpha t}.$$

3. Main results. If we integrate (1.1) from zero to t, we obtain

$$\begin{aligned} x(t;\omega) &= x_0(\omega) + \int_0^t h(\tau, x(\tau;\omega))\,d\tau + \int_0^t \left[\int_\tau^t k(s,\tau;\omega)\,ds\right] f(x(\tau;\omega))\,d\tau \\ &= x_0(\omega) + \int_0^t h(\tau, x(\tau;\omega))\,d\tau + \int_0^t K(t,\tau;\omega) f(x(\tau;\omega))\,d\tau, \end{aligned} \tag{3.1}$$

where $x_0(\omega) = x(0;\omega)$ and $K(t,\tau;\omega)$ is given by (2.3).

We now prove the following existence theorem.

THEOREM 3.1. *Suppose the random equation* (3.1) *satisfies the following conditions*:

(i) *B and D are Banach spaces stronger than C_c and the pair (B,D) is admissible with respect to each of the integral operators*

$$(T_1 x)(t;\omega) = \int_0^t x(\tau;\omega)\,d\tau$$

and

$$(T_2 x)(t;\omega) = \int_0^t K(t,\tau;\omega) x(\tau;\omega)\,d\tau, \qquad t \geqq 0,$$

where $K(t,\tau;\omega)$ behaves as above;

(ii) *$x(t;\omega) \to h(t, x(t;\omega))$ is an operator on*

$$S = \{x(t;\omega) \in D : \|x(t;\omega)\|_D \leqq \rho\}$$

with values in B satisfying

$$\|h(t, x(t;\omega)) - h(t, y(t;\omega))\|_B \leqq \lambda_1 \|x(t;\omega) - y(t;\omega)\|_D$$

for $x(t;\omega)$, $y(t;\omega) \in S$ and λ_1 constant;

(iii) *$x(t;\omega) \to f(x(t;\omega))$ is an operator on S with values in B satisfying $f(0) = 0$ and*

$$\|f(x(t;\omega)) - f(y(t;\omega))\|_B \leqq \lambda_2 \|x(t;\omega) - y(t;\omega)\|_D$$

for $x(t;\omega)$, $y(t;\omega) \in S$ and λ_2 constant;

(iv) $x_0(\omega) \in D$.

Then there exists a unique random solution of (3.1), *$x(t;\omega) \in S$, provided*

$$\lambda_1 K_1 + \lambda_2 K_2 < 1,$$

$$\|x_0(\omega)\|_D + K_1 \|h(t,0)\|_B \leqq \rho(1 - \lambda_1 K_1 - \lambda_2 K_2),$$

where K_1 and K_2 are the norms of T_1 and T_2, respectively.

Proof. By condition (i) and Lemmas 2.1, 2.2, and 2.3, T_1 and T_2 are continuous from B into D. Hence, their norms K_1 and K_2 exist.

Define the operator U from S into D by

$$\begin{aligned}(Ux)(t;\omega) &= x_0(\omega) + (T_1hx)(t;\omega) + (T_2fx)(t;\omega)\\ &= x_0(\omega) + \int_0^t h(\tau, x(\tau;\omega))\,d\tau + \int_0^t K(t,\tau;\omega)f(x(\tau;\omega))\,d\tau.\end{aligned} \tag{3.2}$$

We must show that $U(S) \subset S$ and that U is a contraction operator on S. Then we may apply Banach's fixed-point theorem to obtain the existence of a unique random solution.

Let $x(t;\omega) \in S$. Taking norms in (3.2), we get

$$\begin{aligned}\|(Ux)(t;\omega)\|_D &= \left\|x_0(\omega) + \int_0^t h(\tau, x(\tau;\omega))\,d\tau + \int_0^t K(t,\tau;\omega)f(x(\tau;\omega))\,d\tau\right\|_D\\ &\leqq \|x_0(\omega)\|_D + K_1\|h(t, x(t;\omega))\|_B + K_2\|f(x(t;\omega))\|_B\end{aligned}$$

by the remark following Lemma 2.1. By condition (ii),

$$\|h(t, x(t;\omega))\|_B \leqq \lambda_1\|x(t;\omega) - 0\|_D + \|h(t,0)\|_B,$$

and by condition (iii),

$$\|f(x(t;\omega))\|_B \leqq \lambda_2\|x(t;\omega) - 0\|_D.$$

Hence,

$$\begin{aligned}\|(Ux)(t;\omega)\|_D &\leqq \|x_0(\omega)\|_D + \lambda_1K_1\|x(t;\omega)\|_D + K_1\|h(t,0)\|_B + \lambda_2K_2\|x(t;\omega)\|_D\\ &\leqq \|x_0(\omega)\|_D + K_1\|h(t,0)\|_B + (\lambda_1K_1 + \lambda_2K_2)\rho\\ &\leqq \rho(1 - \lambda_1K_1 - \lambda_2K_2) + (\lambda_1K_1 + \lambda_2K_2)\rho\\ &= \rho,\end{aligned}$$

by the last condition of the theorem. Thus, $U(S) \subset S$.

Let $y(t;\omega)$ be another element of S. We have, since the difference of two elements of a Banach space is in the Banach space,

$$\begin{aligned}\|(Ux)(t;\omega) - (Uy)(t;\omega)\|_D &= \Bigg\|x_0(\omega) + \int_0^t h(\tau, x(\tau;\omega))\,d\tau\\ &\quad + \int_0^t K(t,\tau;\omega)f(x(\tau;\omega))\,d\tau\\ &\quad - x_0(\omega) - \int_0^t h(\tau, y(\tau;\omega))\,d\tau\\ &\quad - \int_0^t K(t,\tau;\omega)f(x(\tau;\omega))\,d\tau\Bigg\|_D\\ &\leqq \left\|\int_0^t [h(\tau, x(\tau;\omega)) - h(\tau, y(\tau;\omega))]\,d\tau\right\|_D\\ &\quad + \left\|\int_0^t K(t,\tau;\omega)[f(x(\tau;\omega)) - f(y(\tau;\omega))]\,d\tau\right\|_D \text{ (cont.)}\end{aligned}$$

$$\leqq K_1\|h(t, x(t;\omega)) - h(t, y(t;\omega))\|_B$$
$$+ K_2\|f(x(t;\omega)) - f(y(t;\omega))\|_B$$
$$\leqq (\lambda_1 K_1 + \lambda_2 K_2)\|x(t;\omega) - y(t;\omega)\|_D,$$

by conditions (ii) and (iii). Since by hypothesis $\lambda_1 K_1 + \lambda_2 K_2 < 1$, U is a contraction operator on S.

Applying Banach's fixed-point theorem, there exists a unique element of S so that

$$(Ux)(t;\omega) = x(t;\omega),$$

that is, there exists a unique random solution of the random equation (1.1), completing the proof.

Now, when the stochastic perturbing term $h(t, x(t;\omega))$ is zero, we obtain a stochastic version of the integro-differential equation studied by Levin [6] as a corollary to Theorem 3.1.

COROLLARY 3.1. *If the stochastic integro-differential equation*

$$x'(t;\omega) = \int_0^t k(t, \tau;\omega) f(x(\tau;\omega))\, d\tau \tag{3.3}$$

satisfies the following conditions:

(i) *B and D are stronger than $C_c(R_+, L_2(\Omega, A, P))$ and (B, D) is admissible with respect to the operator*

$$(Tx)(t;\omega) = \int_0^t K(t, \tau;\omega)x(\tau;\omega)\, d\tau, \qquad t \geqq 0,$$

where $K(t, \tau, \omega)$ is given by equation (2.3) *and behaves as described in* § 2;

(ii) *$x(t;\omega) \to f(x(t;\omega))$ is an operator on*

$$S = \{x(t;\omega) \in D : \|x(t;\omega)\|_D \leqq \rho\}$$

with values in B satisfying $f(0) = 0$ and

$$\|f(x(t;\omega)) - f(y(t;\omega))\|_B \leqq \lambda\|x(t;\omega) - y(t;\omega)\|_D$$

for $x(t;\omega)$, $y(t;\omega) \in S$ and λ constant;

(iii) *$x_0(\omega) \in D$;*

then there exists a unique random solution of equation (3.3), *provided $\lambda K < 1$ and*

$$\|x_0(\omega)\|_D \leqq \rho(1 - \lambda K),$$

where K is the norm of T.

Since (3.3) is the equivalent of (3.1) with $h(t, x)$ equal to zero, the proof follows from that of Theorem 3.1 with T_1 being the null operator.

4. Asymptotic behavior of the random solution. Using the spaces $C_g(R_+, L_2(\Omega, A, P))$ we now give some results concerning the asymptotic behavior of the random solution of (1.1). We first consider the unperturbed equation (3.3).

THEOREM 4.1. *Suppose the equation* (3.3) *satisfies the following conditions*:

(i) $\int_\tau^t |||k(s,\tau;\omega)|||\,ds \leqq \Lambda_1 e^{-\alpha(t-\tau)}$ *for some constants* $\Lambda_1 > 0$ *and* $\alpha > 0$, $0 \leqq \tau \leqq t$;

(ii) $x(t;\omega) \to f(x(t;\omega))$ *satisfies, for some* $\Lambda_2 > 0$ *and* $\alpha > \beta > 0$, $f(0) = 0$,

$$\|f(x(t;\omega))\| \leqq \Lambda_2 e^{-\beta t}, \qquad t \geqq 0,$$

and

$$\|f(x(t;\omega)) - f(y(t;\omega))\| \leqq \lambda \|x(t;\omega) - y(t;\omega)\|$$

for $\|x(t;\omega)\|$ *and* $\|y(t;\omega)\| \leqq \rho e^{-\beta t}$ *at each* $t \geqq 0$ *and* λ *constant*;

(iii) $x_0(\omega) = 0$ P-a.e.

Then there exists a unique random solution of (3.3) *which is stochastically exponentially stable,*

$$\|x(t;\omega)\| = \{E[|x(t;\omega)|^2]\}^{1/2} \leqq \rho e^{-\beta t}, \qquad t \geqq 0,$$

where $E[\cdot]$ *is the mathematical expectation, provided that* λ *is small enough.*

Proof. It is sufficient to show that condition (i) implies the admissibility of the pair of spaces (C_g, C_g) with respect to the operator

$$(Tx)(t;\omega) = \int_0^t K(t,\tau;\omega)x(\tau;\omega)\,d\tau, \qquad t \geqq 0, \tag{4.1}$$

with

$$K(t,\tau;\omega) = \int_\tau^t k(s,\tau;\omega)\,ds, \qquad 0 \leqq \tau \leqq t,$$

and that condition (ii) is equivalent to condition (ii) of Corollary 3.1 with $B = D = C_g(R_+, L_2(\Omega, A, P))$, $g(t) = e^{-\beta t}$, $\beta > 0$.

Taking the norm in $L_2(\Omega, A, P)$ of (4.1), we obtain for $x(t;\omega)$ in the space $C_g(R_+, L_2(\Omega, A, P))$,

$$\begin{aligned}
\|(Tx)(t;\omega)\| &\leqq \int_0^t |||K(t,\tau;\omega)||| \cdot \|x(\tau;\omega)\|\,d\tau \\
&= \int_0^t \left|\left|\left| \int_\tau^t k(s,\tau;\omega)\,ds \right|\right|\right| \cdot \|x(\tau;\omega)\|\,d\tau \\
&\leqq \int_0^t \int_\tau^t |||k(s,\tau;\omega)|||\,ds \|x(\tau;\omega)\|\,d\tau \\
&\leqq \Lambda_1 \int_0^t e^{-\alpha(t-\tau)} \frac{\|x(\tau;\omega)\|}{e^{-\beta\tau}} e^{-\beta\tau}\,d\tau
\end{aligned}$$

by condition (i). But $x(t;\omega) \in C_g(R_+, L_2(\Omega, A, P))$ with $g(t) = e^{-\beta t}$, $t \geqq 0$, $\alpha > \beta > 0$, and we get by definition of the norm in C_g that

$$\begin{aligned}
\|(Tx)(t;\omega)\| &\leqq \Lambda_1 \sup_{t \geqq 0} \left\{ \frac{\|x(t;\omega)\|}{e^{-\beta t}} \right\} e^{-\alpha t} \int_0^t e^{-(\beta-\alpha)\tau}\,d\tau \\
&= \Lambda_1 \|x(t;\omega)\|_{C_g} (\alpha - \beta)^{-1} (e^{-\beta t} - e^{-\alpha t}) \qquad \text{(cont.)}
\end{aligned}$$

$$\leqq \|x(t;\omega)\|_{C_g}\frac{\Lambda_1}{\alpha-\beta}e^{-\beta t}$$

since $\alpha > \beta > 0$. Hence, for $x(t;\omega) \in C_g$, $(Tx)(t,\omega) \in C_g$; that is, (C_g, C_g) is admissible with respect to T.

From condition (ii), $f(x(t;\omega)) \in C_g$, and

$$\sup_{t\geqq 0}\left\{\frac{\|f(x(t;\omega)) - f(y(t;\omega))\|}{e^{-\beta t}}\right\} \leqq \lambda \sup_{t\geqq 0}\left\{\frac{\|x(t;\omega) - y(t;\omega)\|}{e^{-\beta t}}\right\}$$

implies that condition (ii) of Corollary 3.1 holds.

Therefore, by Corollary 3.1, the conclusion follows.

If $h(t, x)$ is not identically equal to zero, then we can still obtain the result that there is a unique random solution of (1.1) which is bounded in the mean square for all $t \in R_+$.

THEOREM 4.2. *Assume that equation* (1.1) *satisfies the following conditions*:

(i) $\int_\tau^t |||k(s,\tau;\omega)|||\,ds \leqq \Lambda_1$ *for some constant* $\Lambda_1 > 0$, $0 \leqq \tau \leqq t$;

(ii) $x(t;\omega) \to h(t, x(t;\omega))$ *satisfies, for some* $\Lambda_2 > 0$ *and* $\beta > 0$,

$$\|h(t, x(t;\omega))\| \leqq \Lambda_2 e^{-\beta t}$$

and

$$\|h(t, x(t;\omega)) - h(t, y(t;\omega))\| \leqq \lambda_1 e^{-\beta t}\|x(t;\omega) - y(t;\omega)\|$$

for $\|x(t;\omega)\|$ *and* $\|y(t;\omega)\| \leqq \rho$, $t \geqq 0$, *and* λ_1 *constant*;

(iii) $x(t;\omega) \to f(x(t;\omega))$ *satisfies* $f(0) = 0$,

$$\|f(x(t;\omega))\| \leqq \Lambda_3 e^{-\beta t}, \qquad \Lambda_3 > 0,$$

and

$$\|f(x(t;\omega)) - f(y(t;\omega))\| \leqq \lambda_2 e^{-\beta t}\|x(t;\omega) - y(t;\omega)\|$$

for $\|x(t;\omega)\|$ *and* $\|y(t;\omega)\| \leqq \rho$, $t \geqq 0$, *and* λ_2 *constant*;

(iv) $x_0(\omega) \in C_1$.

Then there exists a unique random solution of equation (1.1) *satisfying*

$$\|x(t;\omega)\| = \{E[|x(t;\omega)|^2]\}^{1/2} \leqq \rho, \qquad t \in R_+$$

(bounded in the mean square on R_+*), provided that* λ_1, λ_2, $\|x_0(\omega)\|_{C_1}$ *and* $\|h(t,0)\|_{C_g}$ *are sufficiently small.*

Proof. It will suffice to show that the pair of spaces (C_g, C_1) is admissible with respect to the integral operators defined by (2.1), (2.2) and (2.3), under condition (i).

Let $x(t;\omega) \in C_g$. Then from (2.1) we have that

$$\|(T_1x)(t;\omega)\| \leqq \int_0^t \frac{\|x(\tau;\omega)\|}{e^{-\beta\tau}}e^{-\beta\tau}\,d\tau$$

$$\leqq \sup_{t\geqq 0}\left\{\frac{\|x(t;\omega)\|}{e^{-\beta t}}\right\}\int_0^t e^{-\beta\tau}\,d\tau \qquad \text{(cont.)}$$

$$= \|x(t;\omega)\|_{C_g} \frac{1}{\beta}(1 - e^{-\beta t})$$

$$< \|x(t;\omega)\|_{C_g} \frac{1}{\beta} < \infty,$$

by definition of the norm in $C_g(R_+, L_2(\Omega, A, P))$. Hence, $(T_1 x)(t;\omega) \in C_1$, and the pair (C_g, C_1) is admissible with respect to T_1.

Now, from (2.2) and (2.3) for $x(t;\omega) \in C_g$, we obtain

$$\|(T_2 x)(t;\omega)\| \leqq \int_0^t \left|\left|\left| \int_\tau^t k(s,\tau;\omega)\, ds \right|\right|\right| \cdot \|x(\tau;\omega)\|\, d\tau$$

$$\leqq \Lambda_1 \int_0^t \frac{\|x(\tau;\omega)\|}{e^{-\beta\tau}} e^{-\beta\tau}\, d\tau$$

$$\leqq \Lambda_1 \|x(t;\omega)\|_{C_g} \int_0^t e^{-\beta\tau}\, d\tau$$

$$< \Lambda_1 \|x(t;\omega)\|_{C_g} \frac{1}{\beta} < \infty$$

from condition (i). Thus, $(T_2 x)(t;\omega) \in C_1$ and the pair (C_g, C_1) is admissible with respect to T_2.

Therefore, the conditions of Theorem 3.1 hold with $B = C_g$, $g(t) = e^{-\beta t}$, $\beta > 0$, and $D = C_1$, and there exists a unique random solution of (1.1), $x(t;\omega)$, bounded in the mean square by ρ for all $t \in R_+$.

5. Application to a stochastic differential system. Consider the following nonlinear differential system with random parameters:

$$x'(t;\omega) = A(\omega)x(t;\omega) + b(\omega)\phi(\sigma(t;\omega)), \tag{5.1}$$

$$\sigma'(t;\omega) = c^T(t;\omega)x(t;\omega), \tag{5.2}$$

where $A(\omega)$ is an $n \times n$ matrix of measurable functions, $x(t;\omega)$ and $c(t;\omega)$ are $n \times 1$ vectors of random variables for each $t \in R_+$, $b(\omega)$ is an $n \times 1$ vector of measurable functions, $\phi(\sigma)$ is a scalar function, $\sigma(t;\omega)$ is a scalar random variable for each $t \in R_+$, and T denotes the transpose of a matrix.

Note that (5.2) can be written as

$$\sigma(t;\omega) = \sigma(0;\omega) + \int_0^t c^T(s;\omega)x(s;\omega)\, ds,$$

which is similar to a system studied by Tsokos [19].

The system (5.1)–(5.2) may be reduced to a stochastic integro-differential equation of the form (1.1). We may write (5.1) as

$$x(t;\omega) = e^{A(\omega)t}x_0(\omega) + \int_0^t e^{A(\omega)(t-\tau)}b(\omega)\phi(\sigma(\tau;\omega))\, d\tau.$$

Substituting this expression for $x(t;\omega)$ in (5.2), we obtain

$$\sigma'(t;\omega) = c^T(t;\omega)\,e^{A(\omega)t}x_0(\omega) + \int_0^t c^T(t;\omega)\,e^{A(\omega)(t-\tau)}b(\omega)\phi(\sigma(\tau;\omega))\,d\tau.$$

Assume that $|||c^T(t;\omega)||| \leqq K_1$ for all $t \geqq 0$ and $K_1 > 0$ a constant. Also, let $x_0(\omega) \in C_1$, $\phi(0) = 0$, and $b(\omega) \in L_\infty(\Omega, A, P)$. If we assume that the matrix $A(\omega)$ is *stochastically stable*, that is, there exists an $\alpha > 0$ such that

$$P\{\omega : \operatorname{Re}\psi_k(\omega) < -\alpha, k = 1, 2, \cdots, n\} = 1,$$

where $\psi_k(\omega), k = 1, 2, \cdots, n$, are the characteristic roots of the matrix, then it has been shown by Morozan [12] that

$$|||e^{A(\omega)t}||| \leqq K_2\,e^{-\alpha t}$$

for some constant $K_2 > 0$. We also let $\phi(\sigma(t;\omega))$ be in the space C_g with $g(t) = e^{-\alpha t}$, $t \geqq 0$, and

$$|\phi(\sigma_1(t;\omega)) - \phi(\sigma_2(t;\omega))| \leqq \lambda\,e^{-\alpha t}|\sigma_1(t;\omega) - \sigma_2(t;\omega)|.$$

Let

$$h(t, \sigma(t;\omega)) = c^T(t;\omega)\,e^{A(\omega)t}x_0(\omega).$$

Then

$$\begin{aligned}\|h(t, \sigma(t;\omega))\| &\leqq |||c^T(t;\omega)|||K_2\,e^{-\alpha t}\|x_0(\omega)\| \\ &\leqq K_1K_2\,e^{-\alpha t}Z,\end{aligned}$$

where $Z > 0$ is a constant, since $x_0(\omega) \in C_1$. Thus, $h(t, \sigma(t;\omega)) \in C_g$, by definition. Also,

$$\|h(t, \sigma_1(t;\omega)) - h(t, \sigma_2(t;\omega))\| = 0$$

so that it satisfies a Lipschitz condition.

Now, by the assumptions on $c^T(t;\omega)$, $b(\omega)$, and $A(\omega)$, we have

$$k(s, \tau;\omega) = c^T(s;\omega)\,e^{A(\omega)(s-\tau)}b(\omega)$$

satisfying

$$\begin{aligned}\int_\tau^t |||k(s, \tau;\omega)|||\,ds &\leqq \int_\tau^t |||c^T(s;\omega)|||K_2\,e^{-\alpha(s-\tau)}|||b(\omega)|||\,ds \\ &\leqq K_1K_2\,e^{\alpha\tau}|||b(\omega)|||\int_\tau^t e^{-\alpha s}\,ds \\ &= K_1K_2|||b(\omega)|||\frac{1}{\alpha}[1 - e^{-\alpha(t-\tau)}] \\ &= K_1K_2|||b(\omega)|||\frac{1}{\alpha}.\end{aligned}$$

Therefore, all conditions of Theorem 4.2 are satisfied and there exists a unique random solution of the system (5.1)–(5.2) which is bounded in the mean square on R_+.

REFERENCES

[1] M. W. ANDERSON, *Stochastic integral equations*, Doctoral thesis, University of Tennessee, Knoxville, 1966.

[2] A. T. BHARUCHA-REID, *On the theory of random equations*, Proc. Sympos. Applied Math., vol. 16, American Mathematical Society, Providence, 1964, pp. 40–69.

[3] ———, *On random solutions of integral equations in Banach spaces*, Trans. Second Prague Conf. Information Theory, Statistical Decision Functions, and Random Processes, Academic Press, New York, 1960, pp. 27–48.

[4] ———, *Random Integral Equations*, Academic Press, New York, 1972.

[5] C. CORDUNEANU, *Problèmes globaux dans le théorie des équations intégrales de Volterra*, Ann. Mat. Pura Appl., 67 (1965), pp. 349–363.

[6] J. J. LEVIN, *The asymptotic behavior of the solution of a Volterra equation*, Proc. Amer. Math. Soc., 14 (1963), pp. 534–541.

[7] J. J. LEVIN AND J. A. NOHEL, *On a system of integro-differential equations occurring in reactor dynamics*, J. Math. Mech., 9 (1960), pp. 347–368.

[8] ———, *Perturbations of a nonlinear Volterra equation*, Michigan Math. J., 12 (1965), pp. 431–447.

[9] ———, *A system of nonlinear integro-differential equations*, Ibid., 13 (1966), pp. 257–270.

[10] STIG-OLOF LONDON, *On a nonlinear Volterra integro-differential equation*, Commentationes Physico-Mathematicae, 38 (1969), pp. 5–11.

[11] R. K. MILLER, *On Volterra's population equation*, this Journal, 14 (1966), pp. 446–452.

[12] T. MOROZAN, *Stability of linear systems with random parameters*, J. Differential Equations, 3 (1967), pp. 170–178.

[13] J. A. NOHEL, *Qualitative behavior of solutions of nonlinear Volterra equations*, Stability Problems of Differential Equations, Proc. NATO Adv. Study Inst., Padua, Italy, 1965.

[14] ——— *Remarks on nonlinear Volterra equations*, Proc. U.S.–Japan Seminar on Differential and Functional Equations (Minneapolis, Minn., 1967), W. A. Benjamin, New York, 1967, pp. 249–266.

[15] M. N. OĞUZTÖRELI, *Time-Lag Control Systems*, Academic Press, New York, 1966.

[16] W. J. PADGETT AND C. P. TSOKOS, *Random Integral Equations with Applications to Stochastic Systems*, Lecture Notes in Mathematics, vol. 233, Springer-Verlag, Berlin, 1971.

[17] M. R. M. RAO AND C. P. TSOKOS, *Integrodifferential equations of Volterra type*, Bull. Austral. Math. Soc., 3 (1970), pp. 9–22.

[18] C. P. TSOKOS, *On a stochastic integral equation of the Volterra type*, Math. Systems Theory, 3 (1969), pp. 222–231.

[19] ———, *The method of V. M. Popov for differential systems with random parameters*, J. Appl. Probability, 8 (1971), pp. 298–310.

[20] K. YOSIDA, *Functional Analysis*, Springer-Verlag, New York, 1965.