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NORMS OF POSITIVE OPERATORS ON L^p-SPACES

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ABSTRACT. Let $0 \leq T : L^p(Y, \nu) \to L^q(X, \mu)$ be a positive linear operator and let $\|T\|_{p,q}$ denote its operator norm. In this paper a method is given to compute $\|T\|_{p,q}$ exactly or to bound $\|T\|_{p,q}$ from above. As an application the exact norm $\|V\|_{p,q}$ of the Volterra operator $Vf(x) = \int_0^x f(t)dt$ is computed.

1. INTRODUCTION

For $1 \leq p < \infty$ let $L^p[0, 1]$ denote the Banach space of (equivalence classes of) Lebesgue measurable functions on [0,1] with the usual norm $\|f\|_p = (\int_0^1 |f|^p dt)^{1/p}$. For a pair p, q with $1 \leq p, q < \infty$ and a continuous linear operator $T : L^p[0, 1] \to L^q[0, 1]$ the operator norm is defined as usual by

$$\|T\|_{p,q} = \sup\{\|Tf\|_q : \|f\|_p = 1\}. \tag{1-1}$$

Define the Volterra operator $V : L^p[0, 1] \to L^q[0, 1]$ by

$$Vf(x) = \int_0^x f(t)dt. \tag{1-2}$$

The purpose of this note is to show that, for a class of linear operators T between L^p spaces which are positive (i.e., $f \geq 0$ a.e. implies $Tf \geq 0$ a.e.), the problem of computing the exact value of $\|T\|_{p,q}$ can be reduced to showing that a certain nonlinear functional equation has a nonnegative solution. We shall illustrate this by computing the value of $\|V\|_{p,q}$ for V defined by (1-2) above.

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We first state this result. If $1 < p < \infty$ then let p' denote the conjugate exponent of p, i.e. $p' = p/(p-1)$ so that $1/p + 1/p' = 1$. For $\alpha, \beta > 0$ let

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

be the Beta function.

Theorem 1. *If $1 < p, q < \infty$ then the norm $\|V\|_{p,q}$ of the Volterra operator $V : L^p[0, 1] \to L^q[0, 1]$ is*

$$\|V\|_{p,q} = (p')^{\frac{1}{q}} q^{\frac{1}{p'}} (p+q')^{\frac{q-p}{pq}} B\left(\frac{1}{q}, \frac{1}{p'}\right)^{-1} \tag{1-3}$$

In the case $p = q$ this reduces to

$$\|V\|_{p,p} = \frac{p^{\frac{1}{p'}}(p')^{\frac{1}{p}}}{B(\frac{1}{p}, \frac{1}{p'})} \tag{1-4}$$

Special cases of this theorem are known. When $p = q = 2k$ is an even integer, then the result is equivalent to the differential inequality of [H-L-P, §7.6]. This seems to be the only case stated in the literature. The cases that p or q equals 1 or ∞ are elementary. It is easy to see that $\|V\|_{p,\infty} = \|V\|_{1,q} = 1$ for $1 \le p \le \infty$ and $1 \le q \le \infty$. It is also straightforward for $1 < p \le \infty$ and $1 \le q < \infty$ that $\|V\|_{p,1} = (1/(p'+1))^{1/p'}$ and $\|V\|_{\infty,q} = (1/(q+1))^{1/q}$.

The proof of Theorem 1 is based on a general result about compact positive operators between L^p spaces. This theorem in turn will be deduced from a general result about norm attaining linear operators between smooth Banach spaces (see §2 for the exact statement of the result).

In what follows (X, μ) and (Y, ν) will be σ-finite measure spaces. If $T : L^p(Y, \nu) \to L^q(X, \mu)$ is a continuous linear operator we denote by T^* the adjoint operator $T^* : L^{q'}(X, \mu) \to L^{p'}(Y, \nu)$. For any real number x let $\operatorname{sgn}(x)$ be the sign of x (i.e. $\operatorname{sgn}(x) = 1$ for $x > 0$, $= -1$ for $x < 0$ and $= 0$ for $x = 0$). Then for any bounded linear operator $T : L^p(Y, \nu) \to L^q(X, \mu)$ with $1 < p, q < \infty$ we call a function $0 \ne f \in L^p(X, \mu)$ a *critical point* of T if for some real number λ we have

$$T^*(\operatorname{sgn}(Tf)|Tf|^{q-1}) = \lambda \operatorname{sgn}(f)|f|^{p-1} \tag{1-5}$$

(such a function f is at least formally a solution to the Euler-Lagrange equation for the variational problem implicit in the definition of $\|T\|_{p,q}$). In the case that T is positive and $f \ge 0$ a.e. (1-5) takes on the simpler form

$$T^*((Tf)^{q-1}) = \lambda f^{p-1}. \tag{1-6}$$

For future reference we remark that the value of λ in (1-5) and (1-6) is not invariant under rescaling of f. If f is replaced by cf for some $c > 0$ then λ is rescaled to $c^{q-p}\lambda$. Recall that a bounded linear operator $T : X \to Y$ between Banach spaces is called *norm attaining* if, for some $0 \ne f \in X$, we have $\|Tf\|_Y = \|T\|\|f\|_X$. In this case T is said to attain its norm at f. The following theorem will be proved in §2.

Theorem 2. *Let* $1 < p, q < \infty$ *and let* $T : L^p(Y, \nu) \to L^q(X, \mu)$ *be a bounded operator.*

(A) *If* T *attains its norm at* $f \in L^p(X, \mu)$, *then* f *is a critical point of* T (*and so satisfies* (1-5) *for some real* λ).

(B) *If* T *is positive and compact, then* (1-6) *has nonzero solutions. If also any two nonnegative critical points* f_1, f_2 *of* T *differ by a positive multiple, then the norm* $\|T\|_{p,q}$ *is given by*

$$\|T\|_{p,q} = \lambda^{\frac{1}{q}} \|f\|_p^{\frac{p-q}{q}}, \tag{1-7}$$

where $f \neq 0$ *is any nonnegative solution to* (1-6).

In §2 we give an extension of Theorem 2(A) to norm attaining operators between Banach spaces with smooth unit spheres and use this result to prove Theorem 2B. Theorem 2 is closely related to results of Graślewicz [Gr], who shows that if T is positive, $p \geq q$ and (1-6) has a solution $f > 0$ a.e. for $\lambda = 1$, then $\|T\|_{p,q} = 1$. In §4 of this paper we indicate an extension of this result. We prove that if there exists a $0 < f$ a.e. such that

$$T^*(Tf)^{q-1} \leq \lambda f^{p-1}, \tag{1-8}$$

then $\|T\|_{p,p} \leq \lambda^{1/p}$ in case $p = q$ and in case $q < p$ we have $\|T\|_{p,q} \leq \lambda^{1/p} \|Tf\|_q^{1-q/p}$ under the additional hypothesis that $Tf \in L^q$. Inequality (1-8) can be used to prove a classical inequality of Hardy. Another application of this result is a factorization theorem of Maurey about positive linear operators from L^p into L^q.

It is worthwhile remarking that in case $p = q = 2$ the equation (1-5) reduces to the linear equation $T^*Tf = \lambda f$. In this case Theorem 2 is closely related to the fact that in a Hilbert space the norm of a compact operator is the square root of the largest eigenvalue of T^*T.

2. Norm attaining linear operators between smooth Banach spaces

Let E be a Banach space and let E^* denote its dual space. If $f^* \in E^*$ then we denote the value of f^* at $f \in E$ by $f^*(f) = < f, f^* >$. If $0 \neq f \in E$ then $f^* \in E^*$ *norms* f if $\|f^*\| = 1$ and $< f, f^* >= \|f\|$. By the Hahn-Banach theorem there always exist such norming linear functionals. A Banach space E is called *smooth* if for every $0 \neq f \in E$ there exists a unique $f^* \in E^*$ which norms f. Geometrically this is equivalent with the statement that at each point f of the unit sphere of E there is a unique supporting hyperplane. It is well known that E is smooth if and only if the norm is Gâteaux differentiable at all points $0 \neq f \in E$ (see e.g. [B]). If E is a smooth Banach space and $0 \neq f \in E$, then denote by $\Theta_E(f)$ the unique element of E^* that norms f, note $\|\Theta_E(f)\| = 1$. For the basic properties of smooth Banach spaces and the continuity properties of the map $f \mapsto \Theta_E(f)$ we refer to [B, part 3, Chapter 1].

The basic examples of smooth Banach spaces are the spaces $L^p(X, \mu)$ where $1 < p < \infty$. For $0 \neq f \in L^p(X, \mu)$ one can easily show that

$$\Theta_{L^p}(f) = \|f\|_p^{-(p-1)} \operatorname{sgn}(f)|f|^{p-1} \tag{2-1}$$

by considering when equality holds in Hölder's inequality.

The following proposition generalizes part (A) of Theorem 2 to norm attaining operators between smooth Banach spaces.

Proposition. *Let $T : E \to F$ be a bounded linear operator between smooth Banach spaces. If T attains its norm at $0 \neq f \in E$ then there exists a real number α such that*

$$T^*(\Theta_F(Tf)) = \alpha \Theta_E(f) \tag{2-2}$$

and the norm of T is given by

$$\|T\| = \alpha. \tag{2-3}$$

Proof. Define $\Lambda_1, \Lambda_2 \in E^*$ by

$$\Lambda_1(h) = < h, \Theta_E(f) >$$
$$\Lambda_2(h) = \frac{1}{\|T\|} < Th, \Theta_F(Tf) >= \frac{1}{\|T\|} < h, T^*(\Theta_F(Tf)) > .$$

Then $\|\Lambda_1\| = 1$ (since $\|\Theta_E(f)\| = 1$) and $\Lambda_1(f) = \|f\|$, so Λ_1 norms f. Similarly $\|\Theta_F(Tf)\| = 1$ implies that $\|\Lambda_2\| \leq 1$, but using $\|Tf\| = \|T\|\|f\|$ we have $\Lambda_2(f) = \|f\|$. Therefore Λ_2 also norms f. The smoothness of E now implies that $\Lambda_1 = \Lambda_2$. Hence (2-2) holds with $\alpha = \|T\|$ as claimed.

Theorem 2(A) now follows from the following lemma.

Lemma. *If $E = L^p(X, \mu)$, $F = L^q(Y, \nu)$ with $1 < p, q < \infty$ and f is a solution of* (2-2)*, then f is a critical point of f, i.e.,*

$$T^*(\operatorname{sgn}(Tf)|Tf|^{q-1}) = \lambda \operatorname{sgn}(f)|f|^{p-1},$$

where

$$\lambda = \alpha^q \|f\|_p^{q-p}. \tag{2-4}$$

Proof. First we note that if f satisfies (2-2), then we have

$$\|Tf\|_q = \langle Tf, \Theta_F(Tf)\rangle = \langle f, T^*\Theta_F(Tf)\rangle = \langle f, \alpha\Theta_E(f)\rangle = \alpha\|f\|_p.$$

Substitution of (2-1) into (2-2) and multiplication by $\|Tf\|_q^{q-1}$ gives

$$\begin{aligned} T^*(\operatorname{sgn}(Tf)|Tf|^{p-1}) &= \alpha\|Tf\|_q^{q-1}\|f\|_p^{-(p-1)} \operatorname{sgn}(f)|f|^{p-1} \\ &= \alpha^q \|f\|_p^{q-p} \operatorname{sgn}(f)|f|^{p-1}. \end{aligned}$$

This completes the proof of the lemma and of Theorem 2(A).

To prove Theorem 2(B), we first make the observation that if $T : E \to F$ is a compact linear operator and E is reflexive, then T attains its norm (since every bounded sequence in E contains a weakly convergent subsequence and T maps weakly convergent sequences onto norm convergent sequences). Now if T is a positive compact operator from $L^p(X, \mu)$ into $L^q(Y, \nu)$, then T attains its norm at a nonnegative $f \in L^p(X, \mu)$ (simply replace f by $|f|$, if T attains its norm at f). If the additional hypothesis of Theorem 2(B) holds, then any other nonnegative critical point f_0 is a positive multiple of f and therefore T also attains its norm at f_0. Now the proposition and the lemma imply that $\|T\| = \alpha$, where α satisfies (2-4). Hence (1-7) holds. This completes the proof of Theorem 2.

3. The norm of the Volterra operator

In this section we shall prove Theorem 1. We first notice that the adjoint operator of the Volterra operator is given by

$$V^* g(x) = \int_x^1 g(t)dt \text{ a.e.} \tag{3-1}$$

Since $\int_0^x f(t)dt$ and $\int_x^1 g(t)dt$ are absolutely continuous functions, we can assume that $V(f)$, respectively $V^*(g)$, equal these integrals everywhere. From Theorem 2(B) and the rescaling property of λ we prove Theorem 1 by showing that

$$V^*((Vf)^{q-1}) = \lambda f^{p-1} \tag{3-2}$$

has a unique positive solution in $L^p[0, 1]$ normalized so that

$$Vf(1) = \int_0^1 f(t)dt = 1. \tag{3-3}$$

Since Vf is chosen to be absolutely continuous, we see that $V^*((Vf)^{q-1})$ can be chosen to be continuously differentiable on $[0, 1]$. Hence any nonnegative solution of (3-2) can be assumed to be continuously differentiable on $[0, 1]$. Also if f is a nonnegative solution of (3-2) normalized so that (3-3) holds, then Vf is nonnegative and $Vf(1) = 1$ so that Vf is positive on a neighborhood of $x = 1$. From (3-1) we conclude that $V^*((Vf)^{q-1})$ is positive on $[0, 1)$. Hence any nonnegative solution of (3-1) and (3-2) can be assumed to be strictly positive and continuously differentiable on $[0, 1)$. Assume now that f is such a solution of (3-1) satisfying (3-2). Take the derivative on both sides in (3-1) and then multiply both sides by f to get the differential equation

$$-(Vf)^{q-1} f = \lambda(p-1) f^{p-1} f'. \tag{3-4}$$

Using that f is the derivative of Vf, we can integrate both sides to get

$$\frac{1}{q} - \frac{1}{q}(Vf)^q = \frac{\lambda(p-1)}{p} f^p \tag{3-5}$$

since $Vf(1) = 1$ and $f(1) = 0$ by (3-2). To simplify the notation we let $v(x) = Vf(x)$. Then $v(x) > 0$ for $x > 0$, $v'(x) > 0$ for $x < 1$, $v'(1) = 0$

and (3-5) becomes

$$\frac{1}{q}(1 - v(x)^q) = \frac{\lambda(p-1)}{p} v'(x)^p \tag{3-6}$$

or

$$c_{p,q} = \frac{v'(x)}{\sqrt[p]{1 - v(x)^q}}, \tag{3-7}$$

where

$$c_{p,q} = \left(\frac{p}{\lambda q(p-1)}\right)^{\frac{1}{p}}. \tag{3-8}$$

Using $v(0) = 0$ we can integrate (3-7) to get

$$c_{p,q} x = \int_0^{v(x)} \frac{1}{\sqrt[p]{1 - t^q}} dt. \tag{3-9}$$

Putting $x = 1$ in this equation we get

$$c_{p,q} = \int_0^1 \frac{1}{\sqrt[p]{1 - t^q}} dt = \frac{1}{q} B\left(\frac{1}{q}, 1 - \frac{1}{p}\right) = \frac{1}{q} B\left(\frac{1}{q}, \frac{1}{p'}\right). \tag{3-10}$$

The integral in (3-10) was reduced to the Beta function by the change of variable $t = u^{1/q}$. The equations (3-9) and (3-10) uniquely determine the function v and therefore also $f = v'$ and the number λ. This shows that (3-2) and (3-3) have a unique nonnegative solution. Moreover starting with v and λ given by (3-9) and (3-10) one sees by working backwards that $f = v'$ is a nonnegative solution of (3-2) and (3-3). Therefore by Theorem 2(B) the norm of V is given by (1-7). From equations (3-10) and (3-8) we can solve for λ to obtain

$$\lambda^{\frac{1}{q}} = \frac{(p')^{\frac{1}{q}} q^{\frac{p-1}{q}}}{B(\frac{1}{q}, \frac{1}{p'})^{\frac{p}{q}}}. \tag{3-11}$$

In case $p = q$ this shows that $\|V\|_{p,p} = \lambda^{1/p}$, which proves (1-4). In case $p \neq q$ we need to compute $\|v'\|_p = \|f\|_p$. To do this, multiply (3-7) by $\sqrt[p]{1 - v^q}$, raise the result to the power $p - 1$, and then multiply by v' to obtain

$$v'(x)^p = c_{p,q}^{p-1} v'(x)(1 - v(x)^q)^{\frac{p-1}{p}}. \tag{3-12}$$

Using $v(0) = 0$ and $v(1) = 1$ we can integrate (3-12) to obtain

$$\begin{aligned} \|f\|_p^p = \|v'\|_p^p &= c_{p,q}^{p-1} \int_0^1 (1 - t^q)^{\frac{1}{p'}} dt \\ &= c_{p,q}^{p-1} \frac{1}{q} B\left(\frac{1}{q}, \frac{1}{p'} + 1\right) \\ &= c_{p,q}^{p-1} \frac{1}{q} \frac{\frac{1}{p'}}{\frac{1}{q} + \frac{1}{p'}} B\left(\frac{1}{q}, \frac{1}{p'}\right) \\ &= \frac{B(\frac{1}{q}, \frac{1}{p'})^p}{q^{p-1}(p + q')}. \end{aligned} \tag{3-13}$$

(Here we used the identity $B(\alpha, \beta+1) = (\beta/(\alpha+\beta))B(\alpha, \beta)$.) Therefore

$$\|f\|_p^{\frac{p-q}{q}} = \frac{B(\frac{1}{q}, \frac{1}{p'})^{\frac{p-q}{q}}}{q^{\frac{(p-1)(p-q)}{pq}}(p+q')^{\frac{p-q}{pq}}} \tag{3-14}$$

Using (3-11) and (3-14) in formula (1-7) now gives formula (1-3) and the proof of Theorem 1 is complete.

4. Bounds for norms of positive operators

In this section we shall consider a positive operator T acting on a space of (equivalence classes of) measurable functions and give a necessary and sufficient condition for T to define a bounded linear operator from $L^p(Y, \nu)$ into $L^q(X, \mu)$, where $1 < q \le p < \infty$ and obtain a bound for $\|T\|_{p,q}$, similar to (1-7). Let $L^0(X, \mu)$ denote the space of a.e. finite measurable functions on X and let $M(X, \mu)$ denote the space of extended real valued measurable functions on X. For some applications it is useful to assume that T is not already defined on all of L^p. Therefore we shall assume that T is defined on an *ideal* L of measurable functions, i.e., a linear subspace of $L^0(Y, \nu)$ such that if $f \in L$ and $|g| \le |f|$ in L^0, then $g \in L$. By L_+ we denote the collection of nonnegative functions in L. A positive linear operator $T : L \to L^0(X, \mu)$ is called *order continuous* if $0 \le f_n \uparrow f$ a.e. and $f_n, f \in L$ imply that $Tf_n \uparrow Tf$ a.e. We first prove that such operators have "adjoints".

Lemma. *Let L be an ideal of measurable functions on (Y, ν) and let T be a positive order continuous operator from L into $L^0(X, \mu)$. Then there exists an operator $T^t : L^0(X, \mu)_+ \to M(Y, \nu)_+$ such that for all $f \in L_+$ and all $g \in L^0(X, \mu)_+$ we have*

$$\int_X (Tf)g\,d\mu = \int_Y f(T^t g)\,d\nu. \tag{4-1}$$

Proof. Assume first that there exists a function $f_0 > 0$ a.e. in L. Let $g \in L^0(X, \mu)_+$. Then we define $\phi : L_+ \to [0, \infty]$ by $\phi(f) = \int (Tf)g\,d\mu$. Since $Tf_0 < \infty$ a.e. we can find $X_1 \subset X_2 \subset \cdots \uparrow X$ such that for all $n \ge 1$ we have

$$\int_{X_n} (Tf_0)g\,d\mu < \infty.$$

Let $L_{f_0} = \{h : |h| \le cf_0$ for some constant $c\}$ and define $\phi_n : L_{f_0} \to \mathbb{R}$ by

$$\phi_n(h) = \int_{X_n} (Th)g\,d\mu.$$

The order continuity of T now implies (through an application of the Radon–Nikodym theorem) that there exists a function $g_n \in L^1(Y, f_0 d\nu)$ such that for all $h \in L_{f_0}$ we have

$$\phi_n(h) = \int_Y h g_n\,d\nu,$$

see e.g. [Z, Theorem 86.3] . Moreover we can assume that $g_1 \leq g_2 \leq \cdots$ a.e. Let $g_0 = \sup g_n$. An application of the monotone convergence theorem now gives

$$\int_X (Th) g\, d\mu = \int_Y h g_0\, d\nu$$

for all $0 \leq h \in L_{f_0}$. The order continuity of T and another application of the monotone convergence theorem now give

$$\int_X (Tf) g\, d\mu = \int_Y f g_0\, d\nu \tag{4-2}$$

for all $0 \leq f \in L$. If we put $T^t g = g_0$, then (4-2) implies that (4-1) holds in case L contains a strictly positive f_0. In case no such f_0 exists in L, then we can find via Zorn's lemma a maximal disjoint system (f_n) in L^+ and apply the above argument to the restriction of T to the functions $f \in L$ with support in the support Y_n of f_n. In this way, we obtain functions g_n with support in Y_n so that for all such f we have

$$\int_X (Tf) g\, d\mu = \int_{Y_n} f g_n\, d\nu.$$

Now define $T^t g = \sup g_n$ and one can easily verify that in this case (4-1) holds again. This completes the proof of the lemma.

The above lemma allows us to define for any positive operator $T : L \to L^0(X, \mu)$ an adjoint operator T^*. Let $N = \{g \in L^0(X, \mu) : T^t(|g|) \in L^0(Y, \nu)\}$ and define $T^* g = T^t g^+ - T^t g^-$ for $g \in N$. It is easy to see that T^* is a positive linear operator from N into $L^0(Y, \nu)$ such that

$$\int_X (Tf) g\, d\mu = \int_Y f (T^* g)\, d\nu \tag{4-3}$$

holds for all $0 \leq f \in L$ and $0 \leq g \in N$. Observe that in case $T : L^p \to L^q$ is a bounded linear operator and $1 \leq p, q < \infty$ then T^* as defined as above is an extension of the Banach space adjoint. The above construction is motivated by the following example.

Example. Let $T(x, y) \geq 0$ be a $\mu \times \nu$-measurable function on $X \times Y$. Let $L = \{f \in L^0(Y, \nu)$ such that $\int T(x, y)|f(y)|d\nu < \infty$ a.e.$\}$ and define T as the integral operator $Tf(x) = \int_Y T(x, y) f(y) d\nu(y)$ on L. Then one can check (using Tonelli's theorem) that $N = \{g \in L^0(X, \mu)$ such that $\int_Y T(x, y)|g(x)|d\mu < \infty$ a.e.$\}$ and that the operator T^* as defined above is the the integral operator $\int_X T(x, y) g(x) d\mu(x)$.

We now present a Hölder type inequality for positive linear operators. The result is known in ergodic theory (see [K], Lemma 7.4). We include the short proof.

Abstract Hölder inequality. *Let L be an ideal of measurable functions on (Y, ν) and T be a positive operator from L into $L^0(X, \mu)$. If $1 < p < \infty$ and $p' = p/(p-1)$, then we have*

$$T(fg) \le T(f^p)^{\frac{1}{p}} T(g^{p'})^{\frac{1}{p'}} \tag{4-4}$$

for all $0 \le f, g$ with $fg \in L$, $f^p \in L$, and $g^{p'} \in L$.

Proof. For any two positive real numbers x and y we have the inequality $x^{1/p} y^{1/p'} \le \frac{1}{p}x + \frac{1}{p'}y$, so that if $0 \le f, g$ with $fg \in L$, $f^p \in L$ and $g^{p'} \in L$, then for any $\alpha > 0$

$$\begin{aligned} T(fg) &= T\left((\alpha f)\left(\frac{1}{\alpha}\right) g\right) \\ &\le \frac{1}{p} T((\alpha f)^p) + \frac{1}{p'} T\left(\left(\frac{1}{\alpha} g\right)^{p'}\right) \\ &= \frac{1}{p}\alpha^p T(f^p) + \frac{1}{p'}\frac{1}{\alpha^{p'}} T(g^{p'}). \end{aligned} \tag{4-5}$$

Now for each $x \in X$ such that $T(f^p)(x) \ne 0$ choose the number α so that $\alpha^p T(f^p)(x) = \frac{1}{\alpha^{p'}} T(g^{p'})(x)$. Then (4-5) reduces to (4-4).

Theorem 3. *Let L be an ideal of measurable functions on (Y, ν) and let T be a positive order continuous linear operator from L into $L^0(X, \mu)$. Let $1 < q \le p < \infty$ and assume there exists $f_0 \in L$ with $0 < f_0$ a.e. and there exists $\lambda > 0$ such that*

$$T^*(Tf_0)^{q-1} \le \lambda f_0^{p-1}, \tag{4-6}$$

and in case $q < p$ also

$$Tf_0 \in L^q(X, \mu). \tag{4-7}$$

Then T can be extended to a positive linear map from $L^p(Y, \nu)$ into $L^q(X, \mu)$ with

$$\|T\|_{p,q} \le \lambda^{\frac{1}{p}} \|Tf_0\|_q^{1-\frac{q}{p}} \tag{4-8}$$

in case $q < p$ and in case $p = q$

$$\|T\|_{p,p} \le \lambda^{\frac{1}{p}}. \tag{4-9}$$

If also $f_0 \in L^p(Y, \nu)$, then

$$\|T\|_{p,q} \le \lambda^{\frac{1}{q}} \|f_0\|_p^{\frac{p-q}{q}}. \tag{4-10}$$

Proof. Define the positive linear operator $S : L^p(Y, \nu) \to L^0(X, \mu)$ by $Sf = (Tf_0)^{(q-p)/p} \cdot Tf$, note that $S = T$ in case $p = q$. Then it is straightforward to verify that $S^*(h) = T^*((Tf_0)^{(q-p)/p} \cdot h)$. This implies that

$$S^*(Sf_0)^{p-1} = S^*((Tf_0)^{\frac{q(p-1)}{p}}) = T^*(Tf_0)^{q-1} \le \lambda f_0^{p-1},$$

i.e., S satisfies (4-6) with $p = q$. Let $Y_n = \{y \in Y : \frac{1}{n} \le f_0(y) \le n\}$. Then $L^\infty(Y_n, \nu) \subset L$. Let $0 \le u \in L^\infty(Y_n, \nu)$. Then we have

$$\begin{aligned}
\int (Su)^p d\mu &= \int S(u f_0^{-\frac{1}{p'}} f_0^{\frac{1}{p'}})^p d\mu \\
&\le \int S(u^p f_0^{-p+1})(Sf_0)^{\frac{p}{p'}} d\mu \quad \text{(Abstract Hölder inequality)} \\
&= \int u^p f_0^{-p+1} S^*(Sf_0)^{(p-1)} d\nu \\
&\le \int u^p f_0^{-p+1} \lambda f_0^{p-1} d\nu \quad \text{by (4-6)} \\
&= \lambda \|u\|_p^p.
\end{aligned}$$

Hence

$$\|Su\|_p \le \lambda^{\frac{1}{p}} \|u\|_p \tag{4-11}$$

for all $0 \le u \in L^\infty(Y_n, d\nu)$. If $0 \le u \in L$, let $u_n = \min(u, n)\chi_{Y_n}$. Then $u_n \uparrow u$ a.e. and (4-11) holds for each u_n. The order continuity of T and the monotone convergence theorem imply that $\|S\|_{p,p} \le \lambda^{1/p}$. Note that in case $p = q$ this proves (4-9). In case $q < p$ define the multiplication operator M, by $Mh = (Tf_0)^{(p-q)/p} \cdot h$. Then (4-7) implies, by means of Hölder's inequality with $r = p/q$, $r' = p/(p-q)$, that $\|M\|_{p,q} \le \|Tf_0\|^{1-q/p}$. The inequality (4-8) follows now from the factorization $T = MS$. Inequality (4-10) follows from (4-8) by using the inequality $\|Tf_0\|_q \le \|T\|_{p,q}\|f_0\|_p$ and solving for $\|T\|_{p,q}$. This completes the proof of the theorem.

The above theorem is an abstract version of what is called the *Schur test* for boundedness of integral operators (see [H-S] for the case $p = q = 2$ and see [G], Theorem 1.I for the case $1 < q \le p < \infty$).

Corollary. *Let L be an ideal of measurable functions on (Y, ν) and let T be a positive order continuous linear operator from L into $L^0(X, \mu)$. Let $1 < q \le p < \infty$ and assume there exists $f_0 \in L^p(Y, \nu)$ with $0 < f_0$ a.e. and there exists $\lambda > 0$ such that*

$$T^*(Tf_0)^{q-1} = \lambda f_0^{p-1}. \tag{4-12}$$

Then T can be extended to a positive linear map from $L^p(Y, \nu)$ into $L^q(X, \mu)$ with

$$\|T\|_{p,q} = \lambda^{\frac{1}{p}} \|Tf_0\|_q^{1-\frac{q}{p}} = \lambda^{\frac{1}{q}} \|f_0\|_p^{\frac{p-q}{q}} \tag{4-13}$$

and T attains its norm at f_0.

Proof. If we multiply both sides of (4-12) by f_0 and then integrate, we get

$$\int_X (Tf_0)^q d\mu = \lambda \int_Y (f_0)^p d\nu. \tag{4-14}$$

This implies that $Tf_0 \in L^q(X, \mu)$, so that by the above theorem the inequalities (4-8) and (4-10) hold. Equality (4-14) shows that $\|Tf_0\|_q = \lambda^{1/q}\|f_0\|_p^{p/q}$, from which it follows that $\|T\|_{p,q} \geq \lambda^{1/q}\|f_0\|_p^{p/q-1}$. Hence we have equality in (4-10). From this it easily follows that (4-13) holds and that $\|Tf_0\|_q = \|T\|_{p,q}\|f_0\|_p$.

Remark. In the above corollary one could hope that in case $p = q$ the equation (4-12) without the hypothesis $f_0 \in L^p$ still would imply that $\|T\|_{p,p} = \lambda^{1/p}$. Theorem 3 still gives inequality (4-9), but this is all that can be said as we see from the following example. Let $X = Y = [0, \infty)$ with $\mu = \nu$ equal to the Lebesgue measure and define the integral operator T by $Tf(x) = \frac{1}{x}\int_0^x f(t)dt$. An easy computation shows that for $1 < p < \infty$ the equality (4-12) holds for some constant $\lambda = \lambda(\alpha)$, whenever $f_0(y) = y^\alpha$ for all $-1 < \alpha < 0$. One can verify that in this case $\alpha = -1/p$ gives the best upperbound for $\|T\|_{p,p}$, in which case $\lambda = (p/(p-1))^p$. Inequality (4-9) is then the classical Hardy inequality.

We now state a converse to the above theorem, which is essentially due to [G, Theorem 1.II]. For the sake of completeness we supply a proof, which is a simplification of the proof given in [G].

Theorem 4. *Let $0 \leq T : L^p(Y, \nu) \to L^q(X, \mu)$ be a positive linear operator and assume $1 < p, q < \infty$. Then for all λ with $\lambda^{1/q} > \|T\|_{p,q}$ there exists $0 < f_0$ a.e. in $L^p(Y, \nu)$ such that*

$$T^*(Tf_0)^{q-1} \leq \lambda f_0^{p-1}. \tag{4-15}$$

Proof. We can assume that $\|T\|_{p,q} = 1$. Then we assume that $\lambda > 1$. Now define $S : L^p(Y, \nu)_+ \to L^p(Y, \nu)_+$ by means of

$$Sf = (T^*(Tf)^{q-1})^{\frac{1}{p-1}}.$$

Then it is easy to verify that $\|f\|_p \leq 1$ implies that $\|Sf\|_p \leq 1$, also that $0 \leq f_1 \leq f_2$ implies that $Sf_1 \leq Sf_2$ and that $0 \leq f_n \uparrow f$ a.e. in L^p implies that $Sf_n \uparrow Sf$ a.e. Now let $0 < f_1$ a.e. in $L^p(Y, \nu)$ such that $\|f_1\|_p \leq (\lambda-1)/\lambda$. For $n > 1$ we define $f_n = f_1 + \frac{1}{\lambda}Sf_{n-1}$. By induction we verify easily that $f_n \leq f_{n+1}$ and that $\|f_n\|_p \leq 1$ for all n. This implies that there exists f_0 in L^p such that $f_n \uparrow f_0$ a.e. and $\|f_0\|_p \leq 1$. Now $Sf_n \uparrow Sf_0$ implies that $f_0 = f_1 + \frac{1}{\lambda}Sf_0$. Hence $Sf_0 \leq \lambda f_0$, which is equivalent to (4-15) and $f_0 \geq f_1 > 0$ a.e., so that $f_0 > 0$ a.e. and the proof is complete.

Now we present an application of the previous two theorems. The result is due to Maurey [M].

Corollary. *Let $0 \leq T : L^p(Y, \nu) \to L^q(X, \mu)$ be a positive linear operator and assume $1 < q < p < \infty$. Then there exists $0 < g$ a.e. in $L^r(X, \mu)$ with $1/r = 1/q - 1/p$ such that $1/g \cdot T : L^p(Y, \nu) \to L^p(X, \mu)$.*

Proof. From the above theorem it follows that there exists $f_0 \in L^p(Y, \nu)$ such that (4-6) and (4-7) hold. The factorization follows now from the proof of Theorem 3.

We conclude with another application of Theorem 3. An ideal L of measurable functions is called a Banach function space if L is Banach space such that $|g| \leq |f|$ in L implies $\|g\| \leq \|f\|$.

Theorem 5. *Let L be a Banach function space and assume that T and T^* are positive linear operators from L into L. Then T defines a bounded linear operator from L^2 into L^2.*

Proof. Let $S = T^*T$. Then S is a positive operator from L into L, so S is continuous (see [Z]). Let $\lambda > r(S)$, where $r(S)$ denotes the spectral radius of S. From the Neumann series of the resolvent operator $R(\lambda, S) = (\lambda - S)^{-1}$ one sees that for all $0 < g \in L$ we have $f_0 = R(\lambda, S)g \geq \frac{1}{\lambda}g > 0$ and $Sf_0 \leq \lambda f_0$, i.e. $T^*(Tf_0) \leq \lambda f_0$ so (4-6) holds with $p = q = 2$. The conclusion follows now from Theorem 3.

A result for integral operators similar to the above theorem was proved in [S] by completely different methods.

Remark. With some minor modifications of the proofs one can show that Theorems 3 and 4 and their corollaries also hold in case $0 < q \leq 1$.

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