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A NOTE ON ROE'S CHARACTERIZATION OF THE SINE FUNCTION

RALPH HOWARD

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ABSTRACT. Let $f^{(n)}(n, n = 0, \pm 1, \pm 2, \ldots$ be a sequence of complex valued functions on the real line with $(d/dx)f^{(n)} = f^{(n+1)}$ and satisfying inequalities $|f^{(n)}(x)| \leq M_n(1 + |x|)^k$ where as $n \to \infty$ the growth conditions $\lim M_n(1 + e)^{-n} = 0$ and $\lim M_n(1 + e)^{-n} = 0$ hold for all $e > 0$. Then $f^{(0)}(x) = p(x)e^{ix} + q(x)e^{-ix}$ where $p$ and $q$ are polynomials of degree at most $k$.

In his paper [1] J. Roe proves

Theorem (Roe [1]). Let $\{f^{(n)}\}_{n=-\infty}^{\infty}$ be a two way infinite sequence of real valued functions defined on the real line $\mathbb{R}$. Assume $f^{(n+1)}(x) = (d/dx)f^{(n)}(x)$ and that there is a constant $M$ so that $|f^{(n)}(x)| \leq M$ for all $n$ and $x$. Then $f^{(0)}(x) = a \sin(x + \phi)$ for some real constants $a$ and $\phi$.

This gives a rather striking characterization of the sine functions $a \sin(x + \phi)$ in terms of the size of their derivatives and antiderivatives. In this note we show that the bounds $|f^{(n)}(x)| \leq M$ can be relaxed to $|f^{(n)}(x)| \leq M_n(1 + |x|)^{k + \alpha}$ with $0 \leq \alpha < 1$ and where the constants only need to have superexponential growth. More generally:

Theorem. Let $\{f^{(n)}\}_{n=-\infty}^{\infty}$ be a sequence of complex valued functions defined on the real numbers with

\begin{equation}
(f^{(n+1)}(x) = \frac{d}{dx}f^{(n)}(x)
\end{equation}

and so that there are constants $M_n \geq 0$, $\alpha \in [0, 1)$, and a nonnegative integer $k$ satisfying

\begin{equation}
|f^{(n)}(x)| \leq M_n(1 + |x|)^{k + \alpha}.
\end{equation}

If

\begin{equation}
\lim_{n \to \infty} \frac{M_n}{(1 + e)^n} = 0 \quad \text{all } e > 0
\end{equation}

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and

\[
\lim_{n \to \infty} \frac{M_n}{(1 + \varepsilon)^n} = 0 \quad \text{all } \varepsilon > 0,
\]

then

\[
f^{(0)}(x) = p(x)e^{ix} + q(x)e^{-ix}
\]

where \( p(x) \) and \( q(x) \) are polynomials of degree at most \( k \).

Remark 1. The conclusion of the theorem can be sharpened by giving a more precise description of the functions \( f^{(n)} \). If \( k = 0, 1, 2, \ldots \) and \( n = 0, \pm 1, \pm 2, \ldots \) then define

\[
p_{n,k}(x) = (1 + \frac{d}{dx})^n x^k
\]

\[
= \sum_{m=0}^{k} \frac{n(n-1) \cdots (n+m-k+1)}{(k-m)!} k(k-1) \cdots (m+1)x^m
\]

where for negative \( n \) we expand \((1 + d/dx)^n\) formally by use of Taylor's theorem. Then \( p_{n,k}(x) \) is a polynomial of degree \( k \) and

\[
p_{n,k}(x) + p'_{n,k}(x) = \left(1 + \frac{d}{dx}\right)p_{n,k}(x) = p_{n+1,k}(x).
\]

When \( n \geq 0 \) these are Laguerre polynomials. This is because

\[
(-1)^k p_{n,k}(-x) = \left(1 - \frac{d}{dx}\right)^n x^k = e^x \frac{d^n}{dx^n}(x^k e^{-x}).
\]

See for example [0, p. 204]. This last equation implies that if \( \lambda \) is a complex number and

\[
f^{(n)}_{k,\lambda}(x) = \lambda^n p_{n,k}(\lambda x)e^{ix}
\]

then \( \{f^{(n)}_{k,\lambda}\}_{n=-\infty}^{\infty} \) satisfies equation (1). Then if \( \{f^{(n)}\}_{n=-\infty}^{\infty} \) is a sequence of functions satisfying the hypothesis of the theorem then there are complex numbers \( a_0, \ldots, a_k, b_0, \ldots, b_k \) so that

\[
f^{(n)} = \sum_{m=0}^{k} (a_m f^{(n)}_{m,+i} + b_m f^{(n)}_{m,-i})
\]

where \((i)^2 = -1\). The proof of this from the theorem is done by induction on \( k \). The details are left to the reader.

Remark 2. The functions \( f^{(n)}_{k,\lambda}(x) \) just defined satisfy

\[
|f^{(n)}_{k,\lambda}(x)| \leq (k+1)!n^k \max(|\lambda|^n, |\lambda|^{n+k})(1 + |x|)^k e^{\text{Re}(\lambda)} x.
\]

By giving \( \lambda \) pure imaginary values close to \( i \) or \(-i\) we see that there is no obvious weakening of the growth conditions (2), (3), or (4).
Remark 3. It is impossible to replace the interval \((-\infty, \infty)\) by a half infinite interval. The functions \(f^{(n)}(x) = (-1)^n e^{-x}\) on \((0, \infty)\) yield a counterexample. (This observation is due to David Richman.)

Proof of the theorem. Let \(f(x) = f^{(0)}(x)\). The, following [1], we will show the support of the Fourier transform \(\hat{f}\) of \(f\) contained in the set \(\{1, -1\}\). As the integral defining the Fourier transform may diverge, we define it as a distribution, that is as a linear functional on the vector spaces \(\mathcal{S}\) of rapidly decreasing functions on \(R\). Explicitly the value of \(\hat{f}\) on \(\phi \in \mathcal{S}\) is
\[
\langle \hat{f}, \phi \rangle = \langle B, \phi \rangle = \int_{-\infty}^{\infty} f(x) \hat{\phi}(x) \, dx.
\]
Here we follow the notation of [2, Chapter 7].

Suppose it has been shown that the support of \(\hat{f}\) is contained in \(\{1, -1\}\). Then a standard result [2, Theorem 6.25, p. 150] implies there is an \(m \geq 0\) and complex numbers \(a_j, b_j\), \(0 \leq j \leq m\) so that
\[
\hat{f} = \sum_{j=0}^{m} a_j \delta_1^{(j)} + \sum_{j=0}^{m} b_j \delta_{-1}^{(j)}
\]
where \(\delta_1\) (resp. \(\delta_{-1}\)) is the delta function at 1 (resp. at -1) and \(\delta_1^{(j)}\) is the \(j\)th distributional derivative of \(\delta_1\). This Fourier transform can be inverted to give that \(f(x) = f^{(0)}(x)\) has the form given by (5) with \(p(x)\) and \(g(x)\) polynomials (of degree at most \(m\)). But \(|f(x)| \leq |f^{(0)}(x)| \leq M_0 (1 + |x|)^{k+\alpha}\). This implies the polynomials have degree at most \(k\).

This reduces the proof to showing

Lemma 1. The conditions (2) and (3) imply the support of \(\hat{f}\) is disjoint from \((1, \infty)\) and \((-\infty, -1)\).

Lemma 2. The conditions (2) and (4) imply the support of \(\hat{f}\) is disjoint from \((-1, 1)\).

Proof of Lemma 1. We will only show the support of \(\hat{f}\) is disjoint from \((1, \infty)\), the proof for \((-\infty, -1)\) being identical. Let \(\phi\) be a smooth function with its support, \(\text{spt}(\phi)\), in \((1, \infty)\). Then \(\text{spt}(\phi) \subseteq [r, \infty)\) for some \(r > 1\). We now need to show \(\langle \hat{f}, \phi \rangle = 0\). Let \(n \geq 0\) and
\[
\psi_n(t) = \frac{\phi(t)}{(-it)^n}.
\]
This is smooth as \(\phi = 0\) near \(t = 0\). Thus differentiating under the integral gives
\[
\psi_n^{(n)}(x) = \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{\phi(t)}{(-it)^n} e^{-itx} \, dx = \hat{\phi}(x).
\]
So

\[
\langle \hat{f}, \phi \rangle = \langle f, \psi_n \rangle \\
= (-1)^n \langle f^{(n)}, \psi_n \rangle \\
= (-1)^n \int_{-\infty}^{\infty} f^{(n)}(x) \psi_n(x) \, dx.
\]

By (2) this implies

\[
|\langle \hat{f}, \phi \rangle| \leq M_n \int_{-\infty}^{\infty} (1 + |x|^{k+\alpha} |\psi_n(x)|) \, dx.
\]

We now estimate \(|\psi_n(x)|\). First using that \(\text{spt}(\phi) \subseteq [r, \infty)\),

\[
|\psi_n(x)| \leq \int_r^{\infty} \frac{\phi(t)}{t^n} \, dt \leq \frac{1}{r^n} \|\phi\|_{L^1}.
\]

Also if \(x \neq 0\) then integration by parts \((k + 2)\) times yields

\[
|\psi_n(x)| = \left| \int_r^{\infty} \left( \frac{d^{k+2}}{dt^{k+2}} \frac{\phi(t)}{(-it)^n} \right) e^{-ixt} \, dt \right|
\]

\[
\leq \frac{1}{|x|^{k+2}} \int_r^{\infty} \left| \frac{d^{k+2}}{dt^{k+2}} \frac{\phi(t)}{t^n} \right| \, dt
\]

\[
\leq \frac{1}{|x|^{k+2}} \sum_{j=0}^{k+2} \binom{k+2}{j} n(n+1) \cdots (n+j-1) \|\phi^{(k+2-j)}\|_{L^1} \int_r^{\infty} \frac{dt}{t^{n+j}}
\]

\[
= \frac{1}{|x|^{k+2}} \sum_{j=0}^{k+2} \binom{k+2}{j} n(n+1) \cdots (n+j-1) \|\phi^{(k+2-j)}\|_{L^1} \cdot \frac{n+1}{n+j-1} \cdot \frac{r^{n+j-1}}{r^{n-1}}
\]

where \(c_1(k, \phi)\) is a constant depending only on \(k\) and \(\phi\). This can be combined with (8) to give

\[
|\psi_n(x)| \leq \begin{cases} 
1/r^n \|\phi\|_{L^1} & |x| \leq 1, \\
\frac{c_1(k, \phi) n^{k+1}}{|x|^{k+2} r^{n-1}} & |x| > 1.
\end{cases}
\]

Using this in (7) gives an estimate of the form

\[
|\langle \hat{f}, \phi \rangle| \leq c_2(k, \alpha, \phi) \frac{n^{k+1}}{r^{n-1}} M_n.
\]

Let \(\epsilon\) be so that \(1 < 1 + \epsilon < r\). Then for large \(n\)

\[
\frac{n^{k+1}}{r^{n-1}} < \frac{1}{(1 + \epsilon)^n}.
\]
Using this and the condition (3) and (11)

$$|\langle \hat{f}, \phi \rangle| \leq c_2(k, \alpha, \phi) \lim_{n \to \infty} \frac{M_n}{(1 + \varepsilon)^n} = 0.$$ 

Therefore $\langle \hat{f}, \phi \rangle = 0$ for all $\phi$ with support in $(1, \infty)$, i.e., the support of $\hat{f}$ is disjoint from $(1, \infty)$ as claimed.

**Proof of Lemma 2.** The proof is very similar to the proof of Lemma 1. Let $\phi$ be a smooth function with support in $(-1, 1)$. Then for some $r < 1$ the inclusion $\text{spt}(\phi) \subseteq [-r, r]$ holds.

$$\langle \hat{f}, \phi \rangle = (f^{(0)}, \hat{\phi}) = \left( \frac{d^n}{dx^n} f^{(-n)}, \hat{\phi} \right)$$

$$= (-1)^n \left( f^{(-n)}, \hat{\phi} \right)$$

$$= (-1)^n \int_{-\infty}^{\infty} f^{(-n)}(x) \hat{\phi}(n)(x) \, dx.$$ 

By (2) this implies

$$|\langle \hat{f}, \phi \rangle| \leq M_{-n} \int_{-\infty}^{\infty} (1 + |x|)^{k+n} |\phi^{(n)}(x)| \, dx. \quad (12)$$

Differenting under the integral and using $\text{spt}(\phi) \subseteq [-r, r]$

$$\phi^{(n)}(x) = \int_{-r}^{r} (-it)^n \phi(t) e^{-itx} \, dt.$$ 

Thus

$$|\phi^{(n)}(x)| \leq \int_{-r}^{r} |t|^n |\phi(t)| \, dt \leq 2r^n \|\phi\|_{L^1}. \quad (13)$$

Also for $x \neq 0$, integration by parts $(k + 2)$ times and calculations similar to those of inequality (9) yield

$$|\hat{\phi}^{(n)}(x)| = \left| \int_{-r}^{r} \left( \frac{d^{k+2}}{dt^{k+2}} (-it)^n \phi(t) \right) e^{-itx} \, dt \right|$$

$$\leq \frac{1}{|x|^{k+2}} \int_{-r}^{r} \left| \frac{d^{k+2}}{dt^{k+2}} (t^n \phi(t)) \right| \, dt$$

$$\leq \frac{c_3(k, \phi) h^{k+2}}{|x|^{k+2}} r^{n-k-2}.$$ 

Putting the last two estimates together

$$|\phi^{(n)}(x)| \leq \begin{cases} 2r^n \|\phi\|_{L^1} & |x| \leq 1, \\ \frac{c_3(k, \phi) h^{k+2}}{|x|^{k+2}} r^{n-k-2} & |x| > 1. \end{cases} \quad (15)$$

Putting this in (12) gives an estimate

$$|\langle \hat{f}, \phi \rangle| \leq c_4(k, \alpha, \phi) M_{-n} h^{k+2} r^{n-k-2}. \quad (16)$$
The proof is now completed in the same manner as the proof of Lemma 1.

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